MATHEMATICS

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JEE (Main)

Binomial Theorem

"Obvious" is the most dangerous word in mathematics......... Bell, Eric Temple

Binomial expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x + y$, $x^2y + \frac{1}{xy^2}$, $3 - x$, $\sqrt{x^2+1} + \frac{1}{(x^3+1)^{1/3}}$ etc.

Terminology used in binomial theorem :

Factorial notation : $\underline{|n}$ or n! is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3.2.1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note : $n! = n\,.\,(n-1)!$; $n \in N$

Mathematical meaning of ^nC_r : The term ^nC_r denotes number of combinations of r things choosen from n distinct things mathematically, $^nC_r = \frac{n!}{(n-r)!\,r!}$, $n \in N$, $r \in W$, $0 \le r \le n$

Note : Other symbols of of ^nC_r are $\binom{n}{r}$ and C(n, r).

Properties related to ^nC_r :

(i) $^nC_r = {}^nC_{n-r}$

Note : If $^nC_x = {}^nC_y \Rightarrow$ Either $x = y$ or $x + y = n$

(ii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(iv) $^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2} = ... = \frac{n(n-1)(n-2).........(n-(r-1))}{r\,(r-1)(r-2).......2\,.1}$

(v) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

Statement of binomial theorem :

$(a + b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_r\, a^{n-r} b^r + ... + {}^nC_n\, a^0 b^n$

where $n \in N$

or $(a + b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r}b^r$

Note : If we put a = 1 and b = x in the above binomial expansion, then

or $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + ... + {}^nC_r\, x^r + ... + {}^nC_n\, x^n$

or $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$

Regarding Pascal's Triangle, we note the following :

(a) Each row of the triangle begins with 1 and ends with 1.

(b) Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.

Example # 3 : The number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is

(A) 21 (B) 31 (C) 41 (D) 61

Solution : $(1 - 3x + 3x^2 - x^3)^{20} = [(1 - x)^3]^{20} = (1 - x)^{60}$

Therefore number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is **61.**

General term :

$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + \text{..........} + {}^nC_r\, x^{n-r} y^r + \text{.........} + {}^nC_n\, x^0 y^n$

$(r + 1)^{th}$ term is called general term and denoted by T_{r+1}.

$T_{r+1} = {}^nC_r\, x^{n-r} y^r$

Note : The r^{th} term from the end is equal to the $(n - r + 2)^{th}$ term from the begining, i.e. ${}^nC_{n-r+1}\, x^{r-1} y^{n-r+1}$

Example # 4 : Find (i) 28^{th} term of $(5x + 8y)^{30}$ (ii) 7^{th} term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

Solution : (i) $T_{27+1} = {}^{30}C_{27}\,(5x)^{30-27}\,(8y)^{27} = \frac{30\,!}{3\,!\,27\,!}\,(5x)^3\,.\,(8y)^{27}$

(ii) 7th term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

$$T_{6+1} = {}^9C_6 \left(\frac{4x}{5}\right)^{9-6} \left(-\frac{5}{2x}\right)^6 = \frac{9!}{3\,!\,6\,!} \left(\frac{4x}{5}\right)^3 \left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3}$$

Example # 5 : Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution : The general term in the expansion of $\left(9^{1/4} + 8^{1/6}\right)^{1000}$ is

$$T_{r+1} = {}^{1000}C_r \left(9^{\frac{1}{4}}\right)^{1000-r} \left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\; 3^{\frac{1000-r}{2}}\; 2^{\frac{r}{2}}$$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{1000 - r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is 501

Middle term(s) :

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2} + 1\right)^{th}$ terms.

Example # 6 : Find the middle term(s) in the expansion of

(i) $\left(1 - \frac{x^2}{2}\right)^{14}$ (ii) $\left(3a - \frac{a^3}{6}\right)^9$

Case - I When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 ...., m - 1)$
i.e. $T_2 > T_1, T_3 > T_2, ..., T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$
i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ $(r = m + 1, m + 2, ...n)$
i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, ...T_{n+1} < T_n$

Conclusion :

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r < \frac{n+1}{1+\left|\frac{a}{b}\right|}$ $(r = 1, 2, 3,........, m-1, m)$

i.e. $T_2 > T_1, T_3 > T_2, ..., T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r > \frac{n+1}{1+\left|\frac{a}{b}\right|}$ $(r = m + 1, m + 2, ...n)$

i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, ..., T_{n+1} < T_n$

Conclusion :

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note : (i) In any binomial expansion, the middle term(s) has greatest binomial coefficient.
In the expansion of $(a + b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even	1	${}^nC_{n/2}$
Odd	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

(Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put a = b = 1, and proceed as discussed above.

Example # 8 : Find the numerically greatest term in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$.

Solution : Let r^{th} and $(r + 1)^{th}$ be two consecutive terms in the expansion of $(3 - 5x)^{15}$

$T_{r+1} \geq T_r$

${}^{15}C_r\, 3^{15-r} (|-5x|)^r \geq {}^{15}C_{r-1}\, 3^{15-(r-1)} (|-5x|)^{r-1}$

$$\frac{(15)!}{(15-r)!\,r!}\,|-5x| \geq \frac{3.(15)!}{(16-r)!\,(r-1)!}$$

$$5 \,.\, \frac{1}{5}\,(16 - r) \geq 3r$$

$16 - r \geq 3r$

$4r \leq 16$ $r \leq 4$

Self practice problems :

(8) If n is a positive integer, then show that $3^{2n+1} + 2^{n+2}$ is divisible by 7.

(9) What is the remainder when 7^{103} is divided by 25 .

(10) Find the last digit, last two digits and last three digits of the number $(81)^{25}$.

(11) Which number is larger $(1.2)^{4000}$ or 800

Answers : (9) 18 (10) 1, 01, 001 (11) $(1.2)^{4000}$.

Some standard expansions :

(i) Consider the expansion

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\ x^{n-r} y^r = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \ldots\ldots\ldots\ldots + {}^nC_r x^{n-r} y^r + \ldots\ldots\ldots\ldots + {}^nC_n x^0 y^n \quad \ldots.(i)$$

(ii) Now replace $y \to -y$ we get

$$(x-y)^n = \sum_{r=0}^{n} {}^nC_r\ (-1)^r x^{n-r} y^r = {}^nC_0 x^n y^0 - {}^nC_1 x^{n-1} y^1 + \ldots + {}^nC_r (-1)^r x^{n-r} y^r + \ldots + {}^nC_n (-1)^n x^0 y^n \quad \ldots.(ii)$$

(iii) Adding (i) & (ii), we get
$(x+y)^n + (x-y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + \ldots\ldots\ldots]$

(iv) Subtracting (ii) from (i), we get
$(x+y)^n - (x-y)^n = 2[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + \ldots\ldots\ldots]$

Properties of binomial coefficients :

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_rx^r + \ldots\ldots\ldots\ldots + C_nx^n$ ......(1)
where C_r denotes ${}^nC_r$

(1) The sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n
Putting x = 1 in (1)

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n \quad \ldots\ldots(2)$$

or $\displaystyle\sum_{r=0}^{n} {}^nC_r = 2^n$

(2) Again putting x = –1 in (1), we get

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots\ldots + (-1)^n\ {}^nC_n = 0 \quad \ldots\ldots(3)$$

or $\displaystyle\sum_{r=0}^{n} (-1)^r\ {}^nC_r = 0$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.
from (2) and (3)

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots\ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots\ldots\ldots = 2^{n-1}$$

(4) Sum of two consecutive binomial coefficients

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \quad \Rightarrow \quad \text{L.H.S.} = {}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] = \frac{n!}{(n-r)!\,(r-1)!}\ \frac{(n+1)}{r(n-r+1)} = \frac{(n+1)!}{(n-r+1)!\,r!} = {}^{n+1}C_r = \text{R.H.S.}$$

II Method : By Integration

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$.

Integrating both sides, within the limits – 1 to 0.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0} = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ... + C_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\frac{1}{n+1} - 0 = 0 - \left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + ... + (-1)^{n+1}\frac{C_n}{n+1}\right]$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1} \quad \text{Proved}$$

Example # 14 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$, then prove that

(i) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

(ii) $C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2}\,C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1.\ C_0^2 + 3\ .\ C_1^2 + 5.\ C_2^2 + ... + (2n + 1)\ .\ C_n^2\ . = 2n.\ {}^{2n-1}C_n + {}^{2n}C_n$.

Solution : (i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$. ....(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n x^0$ ....(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + ... + C_nx^n)\ (C_0x^n + C_1x^{n-1} + ... + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of x^n,

$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

(ii) From the product of (i) and (ii) comparing coefficients of x^{n-2} or x^{n+2} both sides,

$C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2}\,C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$.

(iii) **I Method : By Summation**

L.H.S. $= 1.\ C_0^2 + 3.\ C_1^2 + 5.\ C_2^2 + ... + (2n + 1)\ C_n^2$.

$$= \sum_{r=0}^{n}(2r+1)\,{}^nC_r^2 = \sum_{r=0}^{n}2.r\ .\ ({}^nC_r)^2 + \sum_{r=0}^{n}({}^nC_r)^2$$

$$= 2\sum_{r=1}^{n}.\ n\ .\ {}^{n-1}C_{r-1}\ {}^nC_r + {}^{2n}C_n$$

$(1 + x)^n = {}^nC_0 + {}^nC_1\ x + {}^nC_2\ x^2 + ...{}^nC_n\ x^n$ ....(i)

$(x + 1)^{n-1} = {}^{n-1}C_0\ x^{n-1} + {}^{n-1}C_1\ x^{n-2} + ...+{}^{n-1}C_{n-1}x^0$ ....(ii)

Multiplying (i) and (ii) and comparing coeffcients of x^n.

${}^{n-1}C_0\ .\ {}^nC_1 + {}^{n-1}C_1\ .\ {}^nC_2 + ... + {}^{n-1}C_{n-1}\ .\ {}^nC_n = {}^{2n-1}C_n$

$$\sum_{r=0}^{n}{}^{n-1}C_{r-1}\ .\ {}^nC_r = {}^{2n-1}C_n$$

Hence, required summation is $2n.\ {}^{2n-1}C_n + {}^{2n}C_n$ = R.H.S.

II Method : By Differentiation

$(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + C_3x^6 + ...+ C_n\ x^{2n}$

Multiplying both sides by x

$x(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + ... + C_nx^{2n+1}$.

Differentiating both sides

$x\ .\ n\ (1 + x^2)^{n-1}\ .\ 2x + (1 + x^2)^n = C_0 + 3.\ C_1x^2 + 5.\ C_2\ x^4 + ...+ (2n + 1)\ C_n\ x^{2n}$......(i)

$(x^2 + 1)^n = C_0\ x^{2n} + C_1\ x^{2n-2} + C_2\ x^{2n-4} + ... + C_n$ ....(ii)

Multiplying (i) & (ii)

$(C_0 + 3C_1x^2 + 5C_2x^4 + ... + (2n + 1)\ C_n\ x^{2n})\ (C_0\ x^{2n} + C_1x^{2n-2} + ... + C_n)$

$= 2n\ x^2\ (1 + x^2)^{2n-1} + (1 + x^2)^{2n}$

comparing coefficient of x^{2n},

$C_0^2 + 3C_1^2 + 5C_2^2 + ...+ (2n + 1)\ C_n^2 = 2n\ .\ {}^{2n-1}C_{n-1} + {}^{2n}C_n$.

$C_0^2 + 3C_1^2 + 5C_2^2 + ...+ (2n + 1)\ C_n^2 = 2n\ .\ {}^{2n-1}C_n + {}^{2n}C_n$. Proved

Multinomial theorem :

As we know the Binomial Theorem –

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r \; x^{n-r} y^r = \sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} \; x^{n-r} y^r$$

putting $n - r = r_1$, $r = r_2$

therefore, $$(x + y)^n = \sum_{r_1+r_2=n} \frac{n!}{r_1!\; r_2!} \; x^{r_1} . y^{r_2}$$

Total number of terms in the expansion of $(x + y)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 = n$ i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n + 1$

In the same fashion we can write the multinomial theorem

$$(x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \frac{n!}{r_1!\, r_2! \ldots r_k!} \; x_1^{r_1} . x_2^{r_2} \ldots x_k^{r_k}$$

Here total number of terms in the expansion of $(x_1 + x_2 + \ldots\ldots\ldots + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + \ldots\ldots\ldots + r_k = n$ i.e. ${}^{n+k-1}C_{k-1}$

Example # 17 : Find the coefficient of $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$

Solution : $$(a - b - c + d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!\, r_2!\, r_3!\, r_4!} \; (a)^{r_1} \; (-b)^{r_2} \; (-c)^{r_3} \; (d)^{r_4}$$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2$, $r_2 = 3$, $r_3 = 4$, $r_4 = 1$

$\therefore$ coeff. of $a^2 b^3 c^4 d$ is $\dfrac{(10)!}{2!\,3!\,4!\,1!} \; (-1)^3 \; (-1)^4 = -12600$

Example # 18 : In the expansion of $\left(1 + x + \dfrac{7}{x}\right)^{11}$, find the term independent of x.

Solution : $$\left(1 + x + \frac{7}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11} \frac{(11)!}{r_1!\, r_2!\, r_3!} \; (1)^{r_1} \; (x)^{r_2} \left(\frac{7}{x}\right)^{r_3}$$

The exponent 11 is to be divided among the base variables 1, x and $\dfrac{7}{x}$ in such a way so that we get x^0.

Therefore, possible set of values of (r_1, r_2, r_3) are (11, 0, 0), (9, 1, 1), (7, 2, 2), (5, 3, 3), (3, 4, 4), (1, 5, 5)

Hence the required term is

$$\frac{(11)!}{(11)!}\,(7^0) + \frac{(11)!}{9!\,1!\,1!}\,7^1 + \frac{(11)!}{7!\,2!\,2!}\,7^2 + \frac{(11)!}{5!\,3!\,3!}\,7^3 + \frac{(11)!}{3!\,4!\,4!}\,7^4 + \frac{(11)!}{1!\,5!\,5!}\,7^5$$

$$= 1 + \frac{(11)!}{9!\,2!} \cdot \frac{2!}{1!\,1!}\,7^1 + \frac{(11)!}{7!\,4!} \cdot \frac{4!}{2!\,2!}\,7^2 + \frac{(11)!}{5!\,6!} \cdot \frac{6!}{3!\,3!}\,7^3$$

$$+ \frac{(11)!}{3!\,8!} \cdot \frac{8!}{4!\,4!}\,7^4 + \frac{(11)!}{1!\,10!} \cdot \frac{(10)!}{5!\,5!}\,7^5$$

$$= 1 + {}^{11}C_2 . {}^2C_1 . 7^1 + {}^{11}C_4 . {}^4C_2 . 7^2 + {}^{11}C_6 . {}^6C_3 . 7^3 + {}^{11}C_8 . {}^8C_4 . 7^4 + {}^{11}C_{10} . {}^{10}C_5 . 7^5$$

$$= 1 + \sum_{r=1}^{5} {}^{11}C_{2r} \; . \; {}^{2r}C_r \; . \; 7^r$$

Example-20 : If x is so small such that its square and higher powers may be neglected, then find the value of

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$$

Solution : $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}$

$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right) = \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right) = 1 - \frac{x}{8} - \frac{19}{12}x = 1 - \frac{41}{24}x$

Self practice problems :

(16) Find the possible set of values of x for which expansion of $(3 - 2x)^{1/2}$ is valid in ascending powers of x.

(17) If $y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\ldots\ldots\ldots$, then find the value of $y^2 + 2y$

(18) The coefficient of x^{100} in $\frac{3-5x}{(1-x)^2}$ is

(A) 100 (B) –57 (C) –197 (D) 53

Answers : (16) $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$ (17) 4 (18) C

20. Find the coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$

21. If in the expansion of $(1+x)^m (1-x)^n$, the coefficients of x and x^2 are 3 and –6 respectively. Then find the value of m.

22. Find the number of terms in the expansion of $(1 + 5\sqrt{2}\,x)^9 + (1 - 5\sqrt{2}\,x)^9$.

23. If the coefficients of second, third and fourth terms in the expansion of $(1 + x)^n$ are in A.P., then find the value of n.

24. If in the expansion of $(1 - x)^{2n-1}$ the coefficient of x^r is denoted by a_r, then prove that $a_{r-1} + a_{2n-r} = 0$

25. Using binomial theorem, prove that $2^{3n} - 7n - 1$ is divisible by 49 where $n \in N$.

26. Using binomial theorem, prove that $3^{2n+2} - 8n - 9$ is divisible by 64, $n \in N$.

27. Prove that $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\frac{1}{4.5}+ ..... \infty = \log_e\left(\frac{4}{e}\right)$.

28. Find the sum of the infinite series $1 + \frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+ ....$

29. Prove that $(x^2 - y^2) + \frac{1}{2!}(x^4 - y^4) + \frac{1}{3!}(x^6 - y^6) + ...$ to $\infty = e^{x^2} - e^{y^2}$

Type (IV) : Very Long Answer Type Questions: **[06 Mark Each]**

30. Find the value of $\sum_{r=0}^{n}(-1)^r\ {}^nC_r\ \frac{1+r\log_e 10}{(1+\log_e 10^n)^r}$.

31. If the coefficient of r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in the expansion of $(1 + x)^{14}$ are in A.P, then find the value of r.

32. If the coefficients of three cosecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42. Find n

33. If 3^{rd}, 4^{th}, 5^{th} and 6^{th} terms in the expansion of $(x + \alpha)^n$ be respectively a, b, c and d then prove that

$$\frac{b^2-ac}{c^2-bd}=\frac{5a}{3c}$$

34. If coefficients of three consecutive terms in the expansion of $(1 + x)^n$ be 76,95 and 76. Then find n.

35. If the 2nd, 3rd and 4th terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, find x, a and n.

36. Sum the series from n = 1 to n = ∞ , whose nth term is

(i) $\frac{1}{(n+1)!}$ (ii) $\frac{1}{(n+2)!}$ (iii) $\frac{1}{(2n-1)!}$

37. Prove that

$$\log_e\left(\frac{m}{n}\right)=2\left[\left(\frac{m-n}{m+n}\right)+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+....\right]$$

38. Prove that

$$\log_e\left(\frac{x+1}{x}\right)=2\left[\frac{1}{(2x+1)}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+....\right]$$

A-12. The co-efficient of x in the expansion of $(1 - 2x^3 + 3x^5)\left(1+\frac{1}{x}\right)^8$ is :

(1) 56 (2) 65 (3) 154 (4) 62

A-13. The term containing x in the expansion of $\left(x^2+\frac{1}{x}\right)^5$ is -

(1) 2^{nd} (2) 3^{rd} (3) 4^{th} (4) 5^{th}

A-14. Given that the term of the expansion $(x^{1/3} - x^{-1/2})^{15}$ which does not contain x is 5 m, where $m \in N$, then m=

(1) 1100 (2) 1010 (3) 1001 (4) none

A-15. The term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^3$ is:

(1) – 3 (2) 0 (3) 1 (4) 3

A-16. The term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$ is-

(1) 3/2 (2) 5/4 (3) 5/2 (4) None of these

A-17. Let the co-efficients of x^n in $(1 + x)^{2n}$ & $(1 + x)^{2n-1}$ be P & Q respectively, then $\left(\frac{P+Q}{Q}\right)^5 =$

(1) 9 (2) 27 (3) 81 (4) none of these

A-18. If $(1 + by)^n = (1+ 8y + 24y^2 + ...)$ where $n \in N$ then the value of b and n are respectively-

(1) 4, 2 (2) 2, – 4 (3) 2, 4 (4) – 2, 4

A-19. The coefficient of x^{52} in the expansion $\sum_{m=0}^{100} {}^{100}C_m\ (x-3)^{100-m}.\ 2^m$ is :

(1) ${}^{100}C_{47}$ (2) ${}^{100}C_{48}$ (3) $-{}^{100}C_{52}$ (4) $-{}^{100}C_{100}$

A-20. The co-efficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + ... + (1 + x)^{30}$ is :

(1) ${}^{51}C_5$ (2) ${}^{9}C_5$ (3) ${}^{31}C_6 - {}^{21}C_6$ (4) ${}^{30}C_5 + {}^{20}C_5$

A-21. The term independent of x in $(1 + x)^m\left(1+\frac{1}{x}\right)^n$ is

(1) ${}^{m-n}C_n$ (2) ${}^{m+n}C_n$ (3) ${}^{m+1}C_n$ (4) ${}^{m+n}C_{n+1}$

A-22. $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)...... (1 + x + x^2 + ... + x^{100})$ when written in the ascending power of x then the highest exponent of x is

(1) 5000 (2) 5030 (3) 5050 (4) 5040

Section (B) : Numerically greatest term, Remainder and Divisibility problems

B-1. The numerically greatest term in the expansion of $(2 + 3x)^9$, when x = 3/2 is

(1) ${}^9C_6 . 2^9 . (3/2)^{12}$ (2) ${}^9C_3 . 2^9 . (3/2)^6$ (3) ${}^9C_5 . 2^9 . (3/2)^{10}$ (4) ${}^9C_4 . 2^9 . (3/2)^8$

B-2. The numerically greatest term in the expansion of $(2x + 5y)^{34}$, when x = 3 & y = 2 is :

(1) T_{21} (2) T_{22} (3) T_{23} (4) T_{24}

B-3. The remainder when 2^{2003} is divided by 17 is :

(1) 1 (2) 2 (3) 8 (4) none of these

C-9. The value of $\binom{50}{0}\binom{50}{1}+\binom{50}{1}\binom{50}{2}+..........+\binom{50}{49}\binom{50}{50}$ is, where ${}^nC_r=\binom{n}{r}$

(1) $\binom{100}{50}$ (2) $\binom{100}{51}$ (3) $\binom{50}{25}$ (4) $\binom{50}{25}^2$

C-10. The value of $\sum_{r=1}^{10} r\cdot\frac{{}^nC_r}{{}^nC_{r-1}}$ is equal to

(1) 5 (2n – 9) (2) 10 n (3) 9 (n – 4) (4) none of these

C-11. The value of the expression $\left(\sum_{r=0}^{10}{}^{10}C_r\right)\left(\sum_{K=0}^{10}(-1)^K\frac{{}^{10}C_K}{2^K}\right)$ is :

(1) 2^{10} (2) 2^{20} (3) 1 (4) 2^5

C-12. In the expansion of $(1 + x)^n\left(1+\frac{1}{x}\right)^n$, the term independent of x is-

(1) $C_0^2+2\,C_1^2+.....+(n+1)\,C_n^2$ (2) $(C_0+C_1+....+C_n)^2$

(3) $C_0^2+C_1^2+.......+C_n^2$ (4) None of these

C-13. If $(1 + x)^n = C_0 + C_1x + C_2x^2 +...+C_n.x^n$ then for n odd, $C_1^2 + C_3^2 + C_5^2 +.....+ C_n^2$ is equal to

(1) 2^{2n-2} (2) 2^n (3) $\frac{(2n)!}{2(n!)^2}$ (4) $\frac{(2n)!}{(n!)^2}$

C-14. If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, the value of $\sum_{r=0}^{n}\frac{n-2r}{{}^nC_r}$ is :

(1) $\frac{n}{2}a_n$ (2) $\frac{1}{4}a_n$ (3) na_n (4) 0

Section (D) : Multinomial Theorem, Binomial Theorem for negative and fractional index

D-1. The coefficient of $a^5 b^4 c^7$ in the expansion of $(bc + ca + ab)^8$ is
(1) 280 (2) 240 (3) 180 (4) 32

D-2. If $|x| < 1$, then the co-efficient of x^n in the expansion of $(1 + x + x^2 + x^3 +.......)^2$ is
(1) n (2) n – 1 (3) n + 2 (4) n + 1

D-3 The coefficient of x^4 in the expression $(1 + 2x + 3x^2 + 4x^3 + ...\text{up to }\infty)^{1/2}$ (where $|x| < 1$) is
(1) 1 (2) 3 (3) 2 (4) 5

Section (E) : Exponential and Logarithmic series

E-1_. Sum of the infinite series

$$\frac{1}{2!}+\frac{1+2}{3!}+\frac{1+2+3}{4!}+.....\text{ to }\infty$$

(1) $\frac{e}{3}$ (2) e (3) $\frac{e}{2}$ (4) none of these

E-2_. The coefficient of x^6 in series e^{2x} is

(1) $\frac{4}{45}$ (2) $\frac{3}{45}$ (3) $\frac{2}{45}$ (4) none of these

4. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

(1) the number of irrational terms is 19 (2) middle term is irrational
(3) the number of rational terms is 2 (4) All of these

5. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$, then :
(1) $a_1 = 20$ (2) $a_2 = 210$ (3) $a_4 = 8085$ (4) All of these

6. $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + ... + a_{15}x^{15}$, then a_{10} equals to :
(1) 99 (2) 101 (3) 100 (4) 110

7. In the expansion of $\left(x^3-\frac{1}{x^2}\right)^n$, $n \in N$, if the sum of the coefficients of x^5 and x^{10} is 0, then n is :
(1) 25 (2) 20 (3) 15 (4) None of these

8. The coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$ is :
(1) 70 (2) 112 (3) 105 (4) 210

9. The term in the expansion of $(2x – 5)^6$ which has greatest binomial coefficient is
(1) T_3 (2) T_4 (3) T_5 (4) T_6

10. The remainder when 7^{98} is divided by 5 is
(1) 4 (2) 0 (3) 2 (4) 3

11. The last three digits of the number $(27)^{27}$ is
(1) 805 (2) 301 (3) 503 (4) 803

12. $7^9 + 9^7$ is divisible by :
(1) 7 (2) 24 (3) 64 (4) 72

13. Let $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$. The greatest value of the integer which divides f(n) for all n is :
(1) 27 (2) 9 (3) 3 (4) None of these

14. Coefficient of x^{n-1} in the expansion of, $(x + 3)^n + (x + 3)^{n-1} (x + 2) + (x + 3)^{n-2} (x + 2)^2 +..... + (x + 2)^n$ is :
(1) ${}^{n+1}C_2(3)$ (2) ${}^{n-1}C_2(5)$ (3) ${}^{n+1}C_2(5)$ (4) ${}^{n}C_2(5)$

15. The term in the expansion of $(2x – 5)^6$ which has greatest numerical coefficient is
(1) T_3, T_4 (2) T_4 (3) T_5, T_6 (4) T_6, T_7

16. Number of elements in set of value of r for which, ${}^{18}C_{r-2} + 2.\ {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ is satisfied :
(1) 4 elements (2) 5 elements (3) 7 elements (4) 10 elements

17. The number of values of 'r' satisfying the equation, ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$ is :
(1) 1 (2) 2 (3) 3 (4) 4

18. The sum $\frac{1}{1!\,(n-1)!}+\frac{1}{2!\,(n-2)!}+......\frac{1}{1!\,(n-1)!}$ is equal to :

(1) $\frac{1}{n!}(2^{n-1}-1)$ (2) $\frac{2}{n!}(2^n-1)$ (3) $\frac{2}{n!}(2^{n-1}-1)$ (4) none

PART - II : COMPREHENSION

Comprehension # 1

Let P be a product given by $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$

and Let $S_1 = a_1 + a_2 + \ldots\ldots\ldots + a_n = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum\sum_{i<j<k} a_i.a_j.a_k$ and so on,

then it can be shown that

$P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots\ldots + S_n$.

1. The coefficient of x^8 in the expression $(2 + x)^2 (3 + x)^3 (4 + x)^4$ must be

(1) 26 (2) 27 (3) 28 (4) 29

2. The coefficient of x^{19} in the expression $(x - 1)(x - 2^2)(x - 3^2) \ldots\ldots\ldots (x - 20^2)$ must be

(1) 2870 (2) 2800 (3) –2870 (4) – 4100

3. The coefficient of x^{98} in the expression of $(x - 1)(x - 2) \ldots\ldots\ldots (x - 100)$ must be

(1) $1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2$

(2) $(1 + 2 + 3 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2)$

(3) $\frac{1}{2}[(1 + 2 + 3 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2)]$

(4) None of these

Comprehension # 2

We know that if ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots, {}^nC_n$ be binomial coefficients, then $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots\ldots + C_n x^n$. Various relations among binomial coefficients can be derived by putting $x = 1, -1, i, \omega \left(\text{where } i = \sqrt{-1}, \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$.

4. The value of ${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \ldots\ldots$ must be

(1) 2i (2) $(1 - i)^n - (1 + i)^n$

(3) $\frac{1}{2}[(1 - i)^n + (1 + i)^n]$ (4) $\frac{1}{2}[(2 - i)^n + (1 - i)^n]$

5. The value of expression $({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \ldots\ldots)^2 + ({}^nC_1 - {}^nC_3 + {}^nC_5 \ldots\ldots)^2$ must be

(1) 2^{2n} (2) 2^n (3) 2^{n^2} (4) None of these

Exercise # 3

PART - I : AIEEE PROBLEMS (LAST 10 YEARS)

1. If ${}^nC_r$ denotes the number of combinations of n things taken r at a time, then the expression ${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$ equals **[AIEEE 2003]**

(1) ${}^{n+2}C_r$ (2) ${}^{n+2}C_{r+1}$ (3) ${}^{n+1}C_r$ (4) ${}^{n+1}C_{r+1}$

2. The number of integral terms in the expansion of $\left(\sqrt{3} + \sqrt[8]{5}\right)^{256}$ is : **[AIEEE 2003]**

(1) 32 (2) 33 (3) 34 (4) 35.

13. The sum of the series ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + ... + {}^{20}C_{10}$ is **[AIEEE 2007 (3, –1), 120]**

(1) $-{}^{20}C_{10}$ (2) $\frac{1}{2}\,{}^{20}C_{10}$ (3) 0 (4) ${}^{20}C_{10}$

14. In the binomial expansion of $(a - b)^n$, $n \geq 5$, the sum of 5th and 6th term is zero, then $\frac{a}{b}$ equals **[AIEEE 2008 (3, –1), 105]**

(1) $\frac{n-4}{5}$ (2) $\frac{5}{n-4}$ (3) $\frac{6}{n-5}$ (4) $\frac{n-5}{6}$

15. **Statement-1 :** $\sum_{r=0}^{n}(r+1)\,{}^nC_r = (n+2)\,2^{n-1}$ **[AIEEE 2008 (3, –1), 105]**

Statement-2 : $\sum_{r=0}^{n}(r+1)\,{}^nC_r\,x^r = (1+x)^n + nx\,(1+x)^{n-1}$

(1) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(2) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(3) Statement-1 is True, Statement-2 is False
(4) Statement-1 is False, Statement-2 is True

16. Let $S_1 = \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\,{}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\,{}^{10}C_j$. **[AIEEE 2009 (4, –1), 144]**

Statement -1 : $S_3 = 55 \times 2^9$.
Statement -2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

(1) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement -1.
(2) Statement-1 is true, Statement-2 is false.
(3) Statement -1 is false, Statement -2 is true.
(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

17. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is : **[AIEEE 2011 (4, –1), 120]**

(1) 144 (2) – 132 (3) – 144 (4) 132

18. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE-2012, (4, –1)/120]**

(1) an irrational number (2) an odd positive integer
(3) an even positive integer (4) a rational number other than positive integers

19. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ is : **[AIEEE - 2013, (4, – ¼) 120]**

(1) 4 (2) 120 (3) 210 (4) 310

Answers

BOARD LEVEL SOLUTIONS

Type (I)

1. $^4C_0(x^3)^4\left(\frac{4}{x}\right)^0 + {}^4C_1(x^3)^3\left(\frac{4}{x}\right)^1 + {}^4C_2(x^3)^2\left(\frac{4}{x}\right)^2$

$+ {}^4C_3(x^3)^1\left(\frac{4}{x}\right)^3 + {}^4C_4(x^3)^0\left(\frac{4}{x}\right)^4$

$\Rightarrow x^{12} + 16x^8 + 96x^4 + 256 + \frac{256}{x^4}$ **Ans.**

2. $^6C_0(ax)^6\left(-\frac{b}{x}\right)^0 + {}^6C_1(ax)^5\left(-\frac{b}{x}\right)^1$

$+ {}^6C_2(ax)^4\left(-\frac{b}{x}\right)^2 + {}^6C_3(ax)^3\left(-\frac{b}{x}\right)^3$

$+ {}^6C_4(ax)^2\left(-\frac{b}{x}\right)^4 + {}^6C_5\, ax\left(-\frac{b}{x}\right)^5$

$+ {}^6C_6(ax)^6\left(-\frac{b}{x}\right)^6$

$= a^6x^6 - 6a^5bx^4 + 15a^4b^2x^2 - 20a^3b^3$

$+ \frac{15a^2b^4}{x^2} - \frac{6ab^5}{x^4} + \frac{b^6}{x^6}$

3. $^4C_0\left(\sqrt{\frac{x}{a}}\right)^4\left(-\sqrt{\frac{a}{x}}\right)^0 + {}^4C_1\left(\sqrt{\frac{x}{a}}\right)^3\left(-\sqrt{\frac{a}{x}}\right)^1$

$+ {}^4C_2\left(\sqrt{\frac{x}{a}}\right)^2\left(-\sqrt{\frac{a}{x}}\right)^2 + {}^4C_3\left(\sqrt{\frac{x}{a}}\right)^1\left(-\sqrt{\frac{a}{x}}\right)^3$

$+ {}^4C_4\left(\sqrt{\frac{x}{a}}\right)^0\left(-\sqrt{\frac{a}{x}}\right)^4$

$= \frac{x^2}{a^2} - \frac{4x}{a} + 6 - 4\frac{a}{x} + \frac{a^2}{x^2}$ **Ans.**

4. $\because$ e^{2x+3}

$= e^{2x}.e^3 = e^3\left[1 + \frac{(2x)}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots\right]$

Thus the coefficient of x^2 in the expansion of

e^{2x+3} is $e^3\frac{2^2}{2!} = 2e^3$

5. We have, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ to ∞

Putting $x = 2$, we get

$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} + \ldots$

$\therefore e^2 = 1 + 2 + 2 + 1\cdot333 + 0.666 + 0.266 + 0.088 + 0.025$

$\therefore e^2 = 7\cdot378$

$\therefore e^2 = 7\cdot4$ (correct to one decimal place)

6. $\log_e(1 + 3x + 2x^2) = \log_e[(1 + 2x)(1 + x)]$

$= \log_e(1 + 2x) + \log_e(1 + x)$

$= \left[2x - \frac{(2x)^2}{2} + \frac{1}{3}(2x)^3 - \frac{1}{4}(2x)^4 + \ldots\infty\right]$

$+ \left[x - \frac{1}{2}(x)^2 + \frac{1}{3}(x)^3 - \frac{1}{4}(x)^4 + \ldots\infty\right]$

$= 3x - \frac{5}{2}x^2 + 3x^3 - \frac{17}{4}x^4 + \ldots\infty$

Type (II)

7. $\Rightarrow$ $^5C_0(1 + x)^5(-x^2)^0 + {}^5C_1(1 + x)^4(-x^2)^1$

$+ {}^5C_2(1 + x)^3(-x^2)^2$

$+ {}^5C_3(1 + x)^2(-x^2)^3 + {}^5C_4(1 + x)^1(-x^2)^4$

$+ {}^5C_5(1 + x)^0(-x^2)^5$

$\Rightarrow (1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)$

$+ 5[1 + 4x + 6x^2 + 4x^3 + x^4](-x^2)$

$+ 10[1 + 3x + 3x^2 + x^3](x^4)$

$+ 10[1 + x^2 + 2x](-x^6) + 5(1 + x)(x^8) + (-x^{10})$

$\Rightarrow (1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)$

$+ [-5x^2 - 20x^3 - 30x^4 - 20x^5 - 5x^6]$

$+ (10x^4 + 30x^5 + 30x^6 + 10x^7)$

$+ [-10x^6 - 10x^8 - 20x^7] + 5x^8 + 5x^9 - x^{10}$

$\Rightarrow -x^{10} + 5x^9 - 5x^8 - 10x^7 + 15x^6 + 11x^5 - 15x^4 - 10x^3 + 5x^2 + 5x + 1$ **Ans.**

8. $\Rightarrow$ $^3C_0(x + 2)^3\left(-\frac{1}{x}\right)^0 + {}^3C_1(x + 2)^2\left(-\frac{1}{x}\right)^1$

$+ {}^3C_2(x + 2)^1\left(-\frac{1}{x}\right)^2 + {}^3C_3(x + 2)^0\left(-\frac{1}{x}\right)^3$

$\Rightarrow [x^3 + 8 + 12x + 6x^2] + 3.[x^2 + 4x + 4].\left(-\frac{1}{x}\right)$

$+ 3(x + 2).\left(\frac{1}{x^2}\right) - \frac{1}{x^3}$

16. As $T_{r+1} = {}^nC_r x^{n-r} y^r$ in $(x + y)^n$

Now consider $\left(x - \frac{7}{x}\right)^{17}$

[on comparing n = 17, r = 10, x = x , y = $\frac{-7}{x}$]

$T_{11} = {}^{17}C_{10}\,(x)^{17-10}\left(-\frac{7}{x}\right)^{10} = {}^{17}C_{10}\,x^7.\frac{(-7)^{10}}{x^{10}}$

$\therefore\ T_{11} = 7^{10}\ {}^{17}C_{10}\,x^{-3}$ **Ans.**

Type (III)

17. $(101)^{100} = (1 + 100)^{100}$

$[(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots. + {}^nC_n x^n]$

Using Binomial theorem

$(1 + 100)^{100} = {}^{100}C_0 + {}^{100}C_1 (100)^1$

$+ {}^{100}C_2 (100)^2 + \ldots\ldots. + {}^{100}C_{100} (100)^{100}$

Now $(1+ 100)^{100} - 1 = 1 + 10^4 + {}^{100}C_2\, 10^4 + \ldots 10^{200} - 1$

$= 10^4\,[1 + {}^{100}C_2 + \ldots\ldots.\ 10^{196}]$

$\because\ 1 + {}^{100}C_2 + \ldots.. + 10^{196}$ is a natural number by the virtue of its being the binomial coefficients.

$= 10^4$ N

$\therefore\ (101)^{100} - 1$ is divisible by 10,000.

18. Consider $17^{1995} + 11^{1995} - 7^{1995}$

$\Rightarrow\ (7 + 10)^{1995} + (1 + 10)^{1995} - 7^{1995}$

$\Rightarrow\ {}^{1995}C_0\,(7)^{1995} + {}^{1995}C_1\,(7)^{1994}\,(10)^1$

$+ \ldots.\ {}^{1995}C_{1995}\,(10)^{1995} + {}^{1995}C_0 + {}^{1995}C_1\,(10)^1$

$+ \ldots..\ {}^{1995}C_{1995}\,(10)^{1995} - 7^{1995}$

$\Rightarrow$ Now ${}^{1995}C_0 + 10$

$[{}^{1995}C_1(7)^{1994} + \ldots\ldots..\ {}^{1995}C_{1995}(10)^{1994}$

$+ {}^{1995}C_1 + \ldots\ldots...\ {}^{1995}C_{1995}\,(10)^{1994}]$

$\Rightarrow\ 1 + 10N\,[\because\ {}^{1995}C_1\,(7)^{1994} + \ldots\ldots.$

${}^{1995}C_{1995}\,(10)^{1994} + {}^{1995}C_1 + \ldots\ldots..\ {}^{1995}C_{1995}\,(10)^{1994}]$

= N(natural number as it is the sum of binomial coefficients)

$\therefore$ Units place is 1 **Ans.**

19. Consider $\left(\frac{1}{2} + \frac{\sqrt{4x+1}}{2}\right)^7$ $[\because\ (x + y)^n = {}^nC_0\,x^n +$

${}^nC_1\,x^{n-1}\,y^1 + {}^nC_2\,x^{n-2}\,y^2 + \ldots\ldots. + {}^nC_n\,y^n]$

$= {}^7C_0\left(\frac{1}{2}\right)^7 + {}^7C_1\left(\frac{1}{2}\right)^6\frac{\sqrt{4x+1}}{2} +$

${}^7C_2\left(\frac{1}{2}\right)^5\left(\frac{\sqrt{4x+1}}{2}\right)^2 + \ldots..+\ {}^7C_7\left(\frac{\sqrt{4x+1}}{2}\right)^7$ ...(i)

Now $\left(\frac{1}{2} - \frac{\sqrt{4x+1}}{2}\right)^7$

$= {}^7C_0\left(\frac{1}{2}\right)^7 + {}^7C_1\left(\frac{1}{2}\right)^6\left(-\frac{\sqrt{4x+1}}{2}\right) +$

${}^7C_2\left(\frac{1}{2}\right)^5\left(-\frac{\sqrt{4x+1}}{2}\right)^2 + \ldots + {}^7C_7\left(-\frac{\sqrt{4x+1}}{2}\right)^7$ ...(ii)

(i) – (ii)

$= 2[{}^7C_1\left(\frac{1}{2}\right)^6\frac{\sqrt{4x+1}}{2} + {}^7C_3\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{4x+1}}{2}\right)^3$

$+ {}^7C_5\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{4x+1}}{2}\right)^5 + {}^7C_7\left(\frac{1}{2}\right)^{\circ}\left(\frac{\sqrt{4x+1}}{2}\right)^7]$

$= 2\sqrt{4x+1}\ [{}^7C_1.\frac{1}{2^7} + {}^7C_3.\frac{1}{2^7}.(4x + 1)$

$+ {}^7C_5.\frac{1}{2^7}(4x + 1)^2 + \frac{(4x+1)^3}{27}]$

$= \frac{1}{2^6}\sqrt{4x+1}\,[{}^7C_1 + {}^7C_3\,(4x + 1)$

$+ {}^7C_5\,(4x + 1)^2 + (4x + 1)^3]$

$\therefore\ \frac{1}{\sqrt{4x+1}}\left\{\left(\frac{1+\sqrt{4x+1}}{2}\right)^7 - \left(\frac{1-\sqrt{4x+1}}{2}\right)^7\right\}$

$= \frac{1}{2^6}\,[{}^7C_1 + {}^7C_3\,(4x + 1) + {}^7C_5\,(4x + 1)^2 + (4x + 1)^3]$

$\therefore$ It is a polynomial of degree 3. **Ans.**

20. Let = x = t^6

$\left(\frac{t^6+1}{t^4-t^2+1} - \frac{t^6-1}{t^6-t^3}\right)^{10}$

$\Rightarrow\ \left[\frac{(t^2+1)(t^4-t^2+1)}{t^4-t^2+1} - \frac{(t^3-1)(t^3+1)}{t^3(t^3-1)}\right]^{10}$

$\Rightarrow\ \left[\frac{t^5+t^3-t^3-1}{t^3}\right]^{10}$

$$= \frac{(2n-1)!}{(2n-1)!(n-1)!}(-1)^{r-1} + \frac{(2n-1)!}{(r-1)!(2n-r)!}(-1)^{2n-r}$$

$$= \frac{(2n-1)!}{(2n-r)!(r-1)!}[(-1)^{r-1} + (-1)^{2n-r}]$$

$$= \frac{(2n-1)!}{(2n-r)!(r-1)!}\left[\frac{-1^r}{-1} + \frac{1}{(-1)^r}\right]$$

$$= \frac{(2n-1)!}{(2n-r)!(r-1)!}[0] = 0 \text{ proved.}$$

25. Consider $2^{3n} - 7n - 1 = (8)^n - 7n - 1$
$= (1 + 7)^n - 7n - 1$

$[\because (1 + x)^n = {}^nC_0 + {}^nC_1x + ... + {}^nC_nx^n]$

$= {}^nC_0 + {}^nC_1(7)^1 + {}^nC_2(7)^2 + ... + {}^nC_n(7)^n - 7n - 1$

$= 1 + 7n + {}^nC_2(7)^2 + {}^nC_3(7)^3 + ... + {}^nC_n(7)^n - 7n - 1$

$= {}^nC_2(7)^2 + {}^nC_3(7)^3 + ... + {}^nC_n(7)^n$

$= 7^2[{}^nC_2 + {}^nC_3 7 + ... + {}^nC_n 7^{n-2}]$

$= 49[{}^nC_2 + {}^nC_3 7 + ... + {}^nC_n 7^{n-2}]$

${}^nC_2 + {}^nC_3.7 + ... + {}^nC_n 7^{n-2} = N$

It is a natural number by the virtue of being a sum of binomial coefficients.

$2^{3n} - 7n - 1 = 49\,N$

$\therefore$ $2^{3n} - 7n - 1$ is divisible by 49. Proved.

26. Consider

$3^{2n+2} - 8n - 9 = (3^2)^{n+1} - 8n - 9 = (9)^{n+1} - 8n - 9$

$= (1 + 8)^{n+1} - 8n - 9$ ... same

$= {}^{n+1}C_0 + {}^{n+1}C_1(8)^1 + {}^{n+1}C_2(8)^2 + ... + {}^{n+1}C_{n+1}(8)^{n+1} - 8n - 9$

$= 1 + (n + 1)\,8 + {}^{n+1}C_2(8)^2 + ... + 8^{n+1} - 8n - 9$

$= 1 + 8n + 8 + {}^{n+1}C_2\,(8)^2 + ... + 8^{n+1} - 8n - 9$

$= {}^{n+1}C_2\,(8)^2 + {}^{n+1}C_3\,(8)^3 + ... + 8^{n+1}$

$= (8)^2\,[{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + ... + 8^{n-1}]$

${}^{n+1}C_2 + {}^{n+1}C_3\,(8) + ... + 8^{n-1} = N$

It is a natural number by the virtue of being a sum of binomial coefficients.

$\therefore$ $3^{2n+2} - 8n - 9 = 64N$

$\therefore$ $3^{2n+2} - 8n - 9$ is divisible by 64

27. L.H.S. $= \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + ...$ to ∞

$$= \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) + ...$$

$$= 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{1}{4} - \frac{1}{4} + \frac{1}{5} + \frac{1}{5} ....$$

$$= 1 - 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} .....\text{to } \infty\right)$$

$$= 2\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} ......\text{to } \infty\right] - 1$$

$$= 2\log_e 2 - 1 = \log_e 4 - \log_e e = \log_e\left(\frac{4}{e}\right) = \text{R.H.S.}$$

28. We have $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ to ∞

Put x = 1, we get $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... \infty$

Put x = – 1, we get $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... \infty$

add both equation, we get

$$e + e^{-1} = 2\left[1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ... \infty\right]$$

Hence $1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ... \infty = \frac{1}{2}(e + e^{-1})$

29. L.H.S. $= \left(x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + ...\right)$

$$- \left(y^2 + \frac{y^4}{2!} + \frac{y^6}{3!} + ...\right)$$

$$= \left(1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + ...\right)$$

$$- \left(1 + y^2 + \frac{(y^2)^2}{2!} + \frac{(y^2)^3}{3!} + ...\right)$$

$$= e^{x^2} - e^{y^2}$$

Type (IV)

30. Let $\log_e 10 = x$

Now $\sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{1 + rx}{(1 + xn)^r}$ $[\because \log a^m = m \log a]$

$$\Rightarrow \sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{1}{(1 + xn)^r} + \sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{rx}{(1 + nx)^r}$$

$$\Rightarrow \sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{r}{(1 + nx)^r}$$

$$+ \frac{x}{1 + nx}\sum_{r=1}^{n}(-1)^r\frac{n}{r}\,{}^{n-1}C_{r-1}\frac{r}{(1 + nx)^{r-1}}$$

$$\left[\text{using } {}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}\right]$$

Now $\frac{v}{vi} = \frac{b^2 - ac}{c^2 - bd} = \frac{x^{2n-6}\alpha^6}{x^{2n-8}\alpha^8} \frac{[({}^nC_3)^2 - {}^nC_2.{}^nC_4]}{[({}^nC_4)^2 - {}^nC_3.{}^nC_5]}$

$$\Rightarrow \frac{x^2}{\alpha^2}\frac{\left[\frac{n!n!}{(n-3)!(n-3)!3!3!} - \frac{n!}{(n-2)!2!}\frac{n!}{(n-4)!4!}\right]}{\left[\frac{n!n!}{(n-4)!(n-4)!4!4!} - \frac{n!}{(n-3)!3!}.\frac{n!}{(n-5)!5!}\right]}$$

$$\Rightarrow \frac{x^2}{\alpha^2}\frac{\frac{1}{2!3!}\frac{n!\,n!}{(n-3)!(n-4)!}\left[\frac{1}{(n-3)3} - \frac{1}{4(n-2)}\right]}{\frac{n!\,n!}{(n-4)!(n-5)!}\left[\frac{1}{(n-4)4} - \frac{1}{5(n-3)}\right]\times\frac{1}{3!\,4!}}$$

$$\Rightarrow \frac{x^2}{\alpha^2}.\frac{4!}{2!}\frac{(n-5)!}{(n-3)!}\left[\frac{4n-8-3n+9}{3.4(n-2).(n-3)}\times\frac{5(n-3)(n-4)}{5n-15-4n+16}\right]$$

$$\Rightarrow \frac{x^2}{\alpha^2}.\frac{4!}{2!}\frac{(n-5)!}{(n-3)!}\left[\frac{(n+1)}{12.(n-2)}\times\frac{4.5(n-4)}{(n+1)}\right]$$

$$= \frac{x^2}{\alpha^2} 20 \frac{(n-4)!}{(n-2)!}$$

34. Let T_r, T_{r+1} and T_{r+2} be the three consecutive terms in the expansion of $(1 + x)^n$

As $T_{r+1} = {}^nC_r\, x^r$ in $[1 + x]^n$

$\therefore\ T_r = {}^nC_{r-1}\, x^{r-1} \Rightarrow T_{r+1} = {}^nC_r\, x^r$

$T_{r+2} = {}^nC_{r+1}\, x^{r+1}$

Now it is given that coefficients of

T_r, T_{r+1} and T_{r+2} are 76, 95, 76 repsectively.

${}^nC_{r-1} = 76$ ...(i)

${}^nC_r = 95$ ...(ii)

${}^nC_{r+1} = 76$ ...(iii)

Now $\frac{ii}{i} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{95}{76}$

$\Rightarrow$ 76n – 76r + 76 = 95 r

$\Rightarrow$ 76(n + 1) = 101 r ...(iv)

$\frac{iii}{ii} = \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1} = \frac{76}{95}$

95n – 95r = 76r + 76

95n – 76 = 101r ...(v)

From (iv) 95n – 76 = 76n + 76

19n = 152

n = 8 **Ans.**

35. As $T_{r+1} = {}^nC_r\ x^{n-r}\, y^r$ in $(x + y)^n$ & consider $(x + a)^n$

$T_2 = {}^nC_1\ x^{n-1}\, a^1 = 240$ (given) ...(i)

$T_3 = {}^nC_2\ x^{n-2}\, a^2 = 720$ ...(ii)

$T_4 = {}^nC_3\ x^{n-3}\, a^3 = 1080$ ...(iii)

$\frac{(ii)}{(i)} \Rightarrow \frac{{}^nC_2}{{}^nC_1}.\frac{x^{n-2}}{x^{n-1}}\ \frac{a^2}{a^1} = 3$

$= \frac{n(n-1)}{2n}.\frac{a}{x} = 3 \Rightarrow (n-1).\frac{a}{x} = 6$ ...(iv)

$\frac{(iii)}{(ii)} \Rightarrow \frac{{}^nC_3}{{}^nC_2}.\frac{x^{n-3}}{x^{n-2}}.\frac{a^3}{a^2} = \frac{1080}{720} = \frac{3}{2}$

$\frac{n(n-1)(n-2).2}{6n(n-1)}\ \frac{a}{x} = \frac{3}{2}$

$(n-2)\ \frac{a}{x} = \frac{9}{2}$ ...(v)

$\frac{(iv)}{(v)}\ \frac{n-1}{n-2} = \frac{6}{9}.2 = \frac{4}{3}$

3n – 3 = 4n – 8

n = 5

From (iv)

$4.\frac{a}{x} = 6 \Rightarrow \frac{a}{x} = \frac{3}{2} \Rightarrow a = \frac{3x}{2}$

Put $a = \frac{3x}{2}$ in (i) $\Rightarrow {}^nC_1\ x^{n-1}\, a^1 = 240$

$5.x^4.\frac{3x}{2} = 240$

$x^5 = 32 \Rightarrow x = 2$

Now $a = \frac{3x}{2} = 3$

$\therefore$ n = 5; a = 3; x = 2 **Ans.**

36.

(i) Sum $= \sum_{n=1}^{\infty} t_n = t_1 + t_2 + t_3 + ...$ to ∞

$= \sum_{n=1}^{\infty}\frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$ to ∞

$= \left[\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}.....\infty\right) - 2\right] = e - 2$

(ii) We have, $t_n = \frac{1}{(n+2)!}$

Sum $= \sum_{n=1}^{\infty}\frac{1}{(n+2)!}$

Advanced Level Problems

OBJECTIVE QUESTIONS

*** Marked Questions may have more than one correct option.**

1. If the sum of the co-efficients in the expansion of $(1 + 2x)^n$ is 6561, then the greatest term in the expansion for $x = 1/2$ is :
(1) 4^{th} (2) 5^{th} (3) 6^{th} (4) none of these

2. The expression, $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$ is a polynomial of degree
(1) 5 (2) 6 (3) 7 (4) 8

3. Co-efficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$ is :
(1) 40 (2) 50 (3) 30 (4) 60

4. Co-efficient of x^{15} in $(1 + x + x^3 + x^4)^n$ is :
(1) $\sum_{r=0}^{5} {}^nC_{15-3r}\,{}^nC_r$ (2) $\sum_{r=0}^{5} {}^nC_{5r}$ (3) $\sum_{r=0}^{5} {}^nC_{3r}$ (4) $\sum_{r=0}^{3} {}^nC_{3-r}\,{}^nC_{5r}$

5. If n is even natural and coefficient of x^r in the expansion of $\frac{(1+x)^n}{1-x}$ is 2^n, $(|x| < 1)$, then –
(1) $r \le n/2$ (2) $r \ge (n-2)/2$ (3) $r \le (n+2)/2$ (4) $r \ge n$

6. The coefficient of x^n in polynomial $(x + {}^{2n+1}C_0)(x + {}^{2n+1}C_1)........(x + {}^{2n+1}C_n)$ is -
(1) $2n + 1$ (2) $2^{2n+1} - 1$ (3) 2^{2n} (4) none of these

7. $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\,{}^rC_p\,2^p\right)$ is equal to -
(1) $4^n - 3^n + 1$ (2) $4^n - 3^n - 1$ (3) $4^n - 3^n + 2$ (4) $4^n - 3^n$

8. ${}^nC_0 - 2.3\,{}^nC_1 + 3.3^2\,{}^nC_2 - 4.3^3\,{}^nC_3 + ... + (-1)^n (n+1)\,{}^nC_n\,3^n$ is equal to
(1) $(-1)^n 2^n\left(\frac{3n}{2}+1\right)$ (2) $2^n\left(n+\frac{3}{2}\right)$ (3) $2^n + 5n\,2^n$ (4) $(-2)^n$.

9. If the sum of the coefficients in the expansion of $(2 + 3cx + c^2x^2)^{12}$ vanishes, then c equals to
(1) –1, 2 (2) 1, 2 (3) 1, –2 (4) –1, –2

10. The term independent of x in the expansion of $(1 + x + 2x^2)\left(3x^2 - \frac{1}{3x^2}\right)^4$ is
(1) 10 (2) 2 (3) 0 (4) 6

11*. Let $a_n = \frac{1000^n}{n!}$ for $n \in N$, then a_n is greatest, when
(1) n = 997 (2) n = 998 (3) n = 999 (4) n = 1000

12. $2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - ...... + (-1)^k\binom{n}{k}\binom{n-k}{0} =$
(1) ${}^nC_k$ (2) ${}^{n+1}C_k$ (3) ${}^{n-1}C_k$ (4) ${}^{n+2}C_k$