<table>
<thead>
<tr>
<th>Content Title</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Trigonometric ratio identities &amp; Equations</td>
<td>01-27</td>
</tr>
<tr>
<td>Exercise</td>
<td></td>
</tr>
<tr>
<td>2. Fundamentals of Mathematics - II</td>
<td>28-38</td>
</tr>
<tr>
<td>Exercise</td>
<td></td>
</tr>
<tr>
<td>3. Straight Line</td>
<td>39-70</td>
</tr>
<tr>
<td>Exercise</td>
<td></td>
</tr>
<tr>
<td>4. Circle</td>
<td>70-92</td>
</tr>
<tr>
<td>Exercise</td>
<td></td>
</tr>
<tr>
<td>Exercise</td>
<td></td>
</tr>
<tr>
<td>6. Solution of Triangle</td>
<td>101-125</td>
</tr>
<tr>
<td>Exercise</td>
<td></td>
</tr>
</tbody>
</table>
TRIGONOMETRIC RATIO, IDENTITIES & EQUATIONS

EXERCISE # 1

PART - I

Section (A) :

A-2.  \( \pi^c = 180^\circ \)

A-4.  
(a)  \( 3 + 2 + 3 \times \frac{1}{3} = 6 \)
(b)  \( 2 \times \frac{1}{2} + 2 \times \frac{1}{2} + 2 \times 4 = 10 \)
(c)  \( \frac{1}{5} + 0 = \frac{1}{5} \)

A-6.  \( \frac{(-\cos \theta) \cos \theta}{\sin \theta \times (-\sin \theta)} = \cot^2 \theta \)

A-9.  \( \tan \theta = -\frac{5}{12} \quad \therefore \quad \frac{3\pi}{2} < \theta < 2\pi \)

\( \Rightarrow \quad \sin \theta = -\frac{5}{13} \quad \text{and} \quad \cot \theta = -\frac{12}{5} \)

\[ \text{LHS} = \frac{-\sin \theta - \cot \theta}{-\cos \theta - \cos \theta} = \frac{\sin \theta + \cot \theta}{2 \cos \theta} = \frac{-\frac{5}{13} - \frac{12}{5}}{-2 \times \frac{13}{5}} = \frac{181}{338} = \text{RHS} \]

Section (B) :

B-4.  \( \text{LHS} = \cos^2 \alpha + \cos (\alpha + \beta) \cdot \cos (\alpha - \beta) = \cos^2 \alpha - \cos^2 \alpha + \sin^2 \beta = \sin^2 \beta = \text{RHS} \)

B-6.  
(i)  \( \frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \frac{1}{2} \sin 2A - \frac{1}{2} \sin 2B \)

(ii)  \( \cot (A + 15^\circ) - \tan (A - 15^\circ) = \frac{\cos(A + 15^\circ) \sin(A - 15^\circ)}{\sin(A + 15^\circ) \cos(A - 15^\circ)} = \frac{2 \sin(A + B) \sin(A - B)}{2 \cos(A + B) \sin(A - B)} = \tan (A + B) \)

B-7.  \( A + B = 45^\circ \)

\[ \Rightarrow \quad \tan(A + B) = \tan(45^\circ) \]

\[ \Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1 \quad \Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1 \]

\[ \Rightarrow \quad (1 + \tan A) (1 + \tan B) = 2 \quad \text{put} \quad A = B = \frac{22.10}{2} \]

\[ \Rightarrow \quad (1 + \tan 22\frac{10}{2})^2 = 2 \quad \Rightarrow \quad \tan 22\frac{10}{2} = \sqrt{2} - 1 \]
Section (C) :

C-1.  \[ \text{LHS} = \left\{ \frac{1 - \tan^2 \left( \frac{\alpha - \pi}{4} \right)}{1 + \tan^2 \left( \frac{\alpha - \pi}{4} \right)} + \cos \frac{\alpha}{2} \cot 4\alpha \right\} \sec \frac{9\alpha}{2} \]
\[ = \left\{ -\cos \left( \frac{\alpha - \pi}{2} \right) + \cos \frac{\alpha}{2} \cot 4\alpha \right\} \sec \frac{9\alpha}{2} = \left\{ -\sin \left( \frac{\alpha}{2} \right) + \frac{\cos \frac{\alpha}{2} \cos 4\alpha}{\sin 4\alpha} \right\} \sec \frac{9\alpha}{2} \]
\[ = \frac{1}{\sin 4\alpha} \left[ \cos 4\alpha \cos \frac{\alpha}{2} - \sin 4\alpha \sin \frac{\alpha}{2} \right] \sec \frac{9\alpha}{2} \]
\[ = \frac{1}{\sin 4\alpha} \times \cos \frac{9\alpha}{2} \cdot \frac{1}{\sec 4\alpha} = \csc 4\alpha = \text{RHS} \]

C-3.  (ii)  \[ \frac{\cos A + \sin A}{\cos A - \sin A} - \frac{\cos A - \sin A}{\cos A + \sin A} = \frac{4\sin A \cos A}{\cos^2 A - \sin^2 A} = \frac{2\sin 2A}{\cos 2A} = 2 \tan 2A \]

C-9  \[ \tan\theta \tan(60^\circ + \theta) \tan(60^\circ - \theta) = \tan 3\theta \]
\[ \text{LHS} = \tan\theta \left( \frac{\sqrt{3} + \tan \theta}{1 - \sqrt{3} \tan \theta} \right) \left( \frac{\sqrt{3} - \tan \theta}{1 + \sqrt{3} \tan \theta} \right) \]
\[ = \tan\theta \left( \frac{3 - \tan^2 \theta}{1 - 3\tan^2 \theta} \right) = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta} = \tan 3\theta \]
\[ \text{Put} \quad \theta = 20^\circ \quad \therefore \tan 20^\circ \]
\[ \therefore \quad \tan 20^\circ \tan 80^\circ \tan 40^\circ = \tan 60^\circ = \sqrt{3} \]

Section (D) :

D-1.  Let \( y = \cos x \cdot \cos \left( \frac{2\pi}{3} + x \right) \cos \left( \frac{2\pi}{3} - x \right) \)
\[ y = \frac{1}{2} \cos x \left[ \cos \frac{4\pi}{3} + \cos 2x \right] \quad \Rightarrow \quad y = \frac{1}{2} \cos x \left[ \frac{-1 + 2\cos 2x}{2} \right] \]
\[ y = \frac{1}{4} \left[ 2\cos 2x \cos x - \cos x \right] \quad \Rightarrow \quad y = \frac{1}{4} \left[ \cos 3x + \cos x - \cos x \right] \]
\[ y = \frac{1}{4} \cos 3x \quad \therefore \quad -1 \leq \cos 3x \leq 1 \]
\[ y_{\min} = \frac{-1}{4} \quad \text{and} \quad y_{\max} = \frac{1}{4} \]

D-3.  (i)  \[ y = 10 \cos^2 x - 6 \sin x \cdot \cos x + 2 \sin^2 x \]
\[ = 5 \left( 1 + \cos 2x \right) - 3 \sin 2x + 1 - \cos 2x \]
\[ = 4 \cos 2x - 3 \sin 2x + 6 \quad \therefore \quad -\sqrt{a^2 + b^2} \leq a \cos \theta + b \sin \theta \leq \sqrt{a^2 + b^2} \]
\[ y_{\max} = 5 + 6 = 11 \]
\[ y_{\min} = -5 + 6 = 1 \]
(ii)  \[ y = 1 + 2 \sin x + 3 \cos^2 x \]
\[ y = 1 + 2 \sin x + 3 - 3 \sin^2 x \]
\[ y = 1 - (3 \sin^2 x - 2 \sin x - 3) \]
\[ y = 1 - 3 \left( \sin^2 x - \frac{2}{3} \sin x + \frac{1}{9} \right) - 1 \]

\[ y = 1 - 3 \left( \sin x - \frac{1}{3} \right)^2 - \frac{10}{9} = -3 \left( \sin x - \frac{1}{3} \right)^2 + \frac{13}{3} \]

\[ y_{max} = \frac{13}{3}, \quad y_{mn} = -3 \left( \frac{16}{9} \right) + \frac{13}{3} = -1 \]

(iii)
\[ y = 3 \cos \left( \theta + \frac{\pi}{3} \right) + 5 \cos \theta + 3 \]

\[ y = 3 \cos \theta \left( \frac{1}{2} - 3 \frac{\sqrt{3}}{2} \sin \theta + 5 \cos \theta + 3 \right) \]

\[ y = \frac{3}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 5 \cos \theta + 3 \]

\[ y = \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3 \]

\[ y_{max} = \sqrt{\frac{169}{4} + \frac{27}{4} + 3} = 7 + 3 = 10 \]

\[ y_{mn} = -\sqrt{\frac{169}{4} + \frac{27}{4} + 3} = -7 + 3 = -4 \]

Section (E):

E-2.
(i) \[ \frac{\cos A \cos \sec A - \sin A \sec A}{\cos A + \sin A} = \frac{\cos^2 A - \sin^2 A}{\sin A \cos A (\cos A + \sin A)} = \frac{\cos A - \sin A}{\cos A \sin A} \]

(ii) \[ \frac{1}{\sec \alpha - \tan \alpha} - \frac{1}{\cos \alpha} = \frac{\cos \alpha}{1 - \cos \alpha} = \frac{\cos \alpha}{1 - \sin \alpha} - \frac{1}{(1 - \sin \alpha) \cos \alpha} = \frac{\cos^2 \alpha - 1 + \sin \alpha}{(1 - \sin \alpha) \cos \alpha} \]

\[ = \frac{\sin \alpha - \sin^2 \alpha}{(1 - \sin \alpha) \cos \alpha} = \frac{\sin \alpha}{\cos \alpha} \]

\[\frac{1}{\cos \alpha} \frac{\cos \alpha}{\sin \alpha + 1} = \frac{\sin \alpha + 1 - \cos^2 \alpha}{(1 + \sin \alpha) \cos \alpha} = \frac{\sin \alpha(1 + \sin \alpha)}{(1 + \sin \alpha) \cos \alpha} = \frac{\sin \alpha}{\cos \alpha} \]

(iii) \[ \frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \frac{\cos^3 - \sin^3 A}{\cos A - \sin A} = \cos^2 A + \sin^2 A - \sin A \cos A + \cos^2 A + \sin^2 A + \sin A \cos A = 2 \]

Section (F):

F-1.
(i) \[ \text{LHS} = \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \cos \frac{8\pi}{7} = \cos \frac{\pi}{7} \cos \frac{4\pi}{7} = -\frac{\sin \frac{8\pi}{7}}{2^{\frac{3}{2}} \sin \frac{\pi}{7}} = \frac{1}{8} = \text{RHS} \]

(ii) \[ \text{LHS} = \cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{3\pi}{11} \cos \frac{8\pi}{11} \cos \frac{16\pi}{11} \]

\[ = \cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{4\pi}{11} \cos \frac{8\pi}{11} \cos \frac{16\pi}{11} \]

\[ = \frac{\sin \frac{32\pi}{11}}{2^{\frac{3}{2}} \sin \frac{\pi}{11}} = \frac{1}{32} = \text{RHS} \]
F-2. \[
\text{LHS} = \sin^2 \theta + \sin^2 2\theta + \sin^2 3\theta + \ldots + \sin^2 n\theta \\
= \left( \frac{1 - \cos 2\theta}{2} \right) + \left( \frac{1 - \cos 4\theta}{2} \right) + \ldots + \left( \frac{1 - \cos 2n\theta}{2} \right) \\
= \frac{n - 1}{2} \left[ (\cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos 2n\theta) \right] \\
= \frac{n}{2} \left[ \frac{\sin n(2\theta)}{\sin 2\theta} - \cos \frac{(2\theta + 2n\theta)}{2} \right] = \frac{n}{2} \left[ \frac{\sin n\theta \cdot \cos (n+1)\theta}{\sin \theta} \right] = \text{RHS}
\]

F-7. \[
\cos (S - A) + \cos (S - B) + \cos (S - C) + \cos S \\
= 2 \cos \frac{2S - (A + B)}{2} \cos \frac{B - A}{2} + 2 \cos \frac{2S - C}{2} \cos \frac{-C}{2} \\
= 2 \cos \frac{C}{2} \cos \frac{B - A}{2} + 2 \cos \frac{A + B}{2} \cos \frac{C}{2} = 2 \cos \frac{C}{2} \left( 2 \cos \frac{A}{2} \cos \frac{B}{2} \right)
\]

Section (G):

G-4. \[
\sin 2\theta = \cos 3\theta \quad \Rightarrow \quad \cos \left( \frac{\pi}{2} - 2\theta \right) = \cos 3\theta \\
\Rightarrow \quad \frac{\pi}{2} - 2\theta = 2n\pi \pm 30 \quad \Rightarrow \quad \frac{\pi}{2} - 2\theta = 2n\pi \\
\Rightarrow \quad \theta = 2n\pi - \frac{\pi}{2}, \quad \frac{\pi}{2} - 2n\pi = \frac{\pi}{5} \left( \frac{1}{2} - 2n \right)
\]

G-8. \[
\tan 2\theta \tan \theta = 1 \quad \Rightarrow \quad \sin 2\theta \sin \theta = \cos 2\theta \cos \theta \\
\Rightarrow \quad 0 = \cos 3\theta \quad \Rightarrow \quad 3\theta = (2n + 1) \frac{\pi}{2} \Rightarrow \theta = (2n + 1) \frac{\pi}{6}.
\]

Section (H):

H-4. \[
\cos^2 x + \cos^2 2x + \cos^2 3x = 1 \\
\Rightarrow \quad \frac{1 + \cos 2x}{2} + \frac{1 + \cos 4x}{2} + \frac{1 + \cos 6x}{2} = 1 \quad \Rightarrow \quad \cos 2x + \cos 4x + \cos 6x = -1 \\
\Rightarrow \quad 2\cos 4x \cos 2x = -2\cos^2 2x \quad \Rightarrow \quad \cos 2x = 0 \quad \text{or} \quad \cos 4x + \cos 2x = 0 \\
\Rightarrow \quad 2x = (2n + 1) \frac{\pi}{2} \quad \text{or} \quad 2\cos 3x \cos x = 0 \quad \Rightarrow \quad x = (2n + 1) \frac{\pi}{4}, (2n + 1) \frac{\pi}{6}, (2n + 1) \frac{\pi}{2}
\]

Now \( x = (2n + 1) \frac{\pi}{6} = \frac{n\pi}{3} + \frac{\pi}{6} \) may also be written as

\[ x = (3k + 1) \frac{\pi}{3} + \frac{\pi}{6}, (3k + 2) \frac{\pi}{3} + \frac{\pi}{6}, (3k) \frac{\pi}{3} + \frac{\pi}{6} \]

\[ = k\pi + \frac{\pi}{2}, \quad k\pi + \frac{5\pi}{6}, \quad k\pi + \frac{\pi}{6} \]

\[ = (k + 1) \frac{\pi}{2}, \quad k\pi + \frac{\pi}{6} \]

\( (k\pi + \frac{\pi}{2} \text{ is same as } (2n + 1) \frac{\pi}{2}) = m\pi \pm \frac{\pi}{6} \)
H-5. \( \sin^2 n\theta - \sin^2 (n - 1)\theta = \sin^2 \theta \)
\[ \Rightarrow \sin (2n - 1) \sin \theta = \sin^2 \theta \]
\( \Rightarrow \quad \theta = m\pi, \quad \sin (2n - 1) \theta - \sin \theta = 0 \)
\( \Rightarrow \quad 2 \cos n\theta \sin \left(\frac{2n - 2}{2}\right) \theta = 0 \)
\( \Rightarrow \quad n\theta = (2p + 1) \frac{\pi}{2}, \quad (n - 1) \theta = \lambda \pi \)
\( \Rightarrow \quad \theta = m\pi, \quad \frac{\lambda \pi}{n - 1}, \quad \left( p + \frac{1}{2} \right) \frac{\pi}{n} \)

Section (I):

I-1. \( \tan^2 \theta - (1 + \sqrt{3}) \tan \theta + \sqrt{3} = 0 \)
\[ \Rightarrow \quad \tan \theta = 1, \quad \sqrt{3} \quad \Rightarrow \quad \theta = n\pi + \frac{\pi}{4}, \quad le n\pi + \frac{\pi}{3}. \]

I-3. \( 4 \cos \theta - 3 \sec \theta = 2 \tan \theta \)
\[ \Rightarrow \quad 4 \cos^2 \theta - 3 = 2 \sin \theta \]
\[ \Rightarrow \quad 4 \cos^2 \theta - 4 \sin \theta - 3 = 2 \sin \theta \]
\[ \Rightarrow \quad 4 \sin \theta = 8 \frac{1}{2} \cos^2 \theta = \sin \theta \]
\[ \Rightarrow \quad \sin \theta = \cos \frac{\pi}{6} \quad \Rightarrow \quad \sin \theta = \sin \frac{\pi}{10} \quad \Rightarrow \quad \sin \theta = \sin \frac{\pi}{10} \quad \Rightarrow \quad \theta = n\pi + (-1)^n \frac{\pi}{10} \quad \text{or} \quad \theta = n\pi - (-1)^n \frac{3\pi}{10} \]

Section (J):

J-1. \( \sqrt{3} \sin \theta - \cos \theta = \sqrt{2} \)
\[ \Rightarrow \quad 2 \sin \theta = \frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta \]
\[ \Rightarrow \quad 2 \sin \left( \theta - \frac{\pi}{6} \right) = \sqrt{2} \]
\[ \Rightarrow \quad \theta = \frac{\pi}{4} \quad \Rightarrow \quad \theta = \frac{\pi}{2} + n\pi \quad \text{or} \quad \theta = \theta + \frac{\pi}{2} \]

J-2. \( 5 \sin \theta + 2 \cos \theta = 5 \)
\[ \Rightarrow \quad \frac{5}{\sqrt{29}} \sin \theta + \frac{2}{\sqrt{29}} \cos \theta = \frac{5}{\sqrt{29}} \]
\[ \Rightarrow \quad \sin \phi \sin \theta + \cos \phi \cos \theta = \frac{5}{\sqrt{29}} \]
\[ \Rightarrow \quad \cos (\theta - \phi) = \sin \phi = \cos \left( \frac{\pi}{2} - \phi \right) \]
\[ \Rightarrow \quad \theta - \phi = 2n\pi \pm \left( \frac{\pi}{2} - \phi \right) \]
\[ \Rightarrow \quad 0 = 2n\pi + \frac{\pi}{2} \mp \phi + \phi \]
\[ \Rightarrow \quad 0 = 2n\pi + \frac{\pi}{2} \quad 2n\pi - \frac{\pi}{2} + 2\phi \]
For \( \theta = 2n\pi - \frac{\pi}{2} + 2\phi. \)
We have \( \theta = 2n\pi + 2 \left( \phi - \frac{\pi}{4} \right) = 2n\pi + 2 \left( \tan^{-1} \frac{5}{2} - \tan^{-1} 1 \right) \)

\[
\theta = 2n\pi + 2 \tan^{-1} \left( \frac{\frac{5}{2} - 1}{1 + \frac{5}{2}} \right) = 2n\pi + 2 \tan^{-1} \left( \frac{3}{7} \right)
\]

\[ \therefore \theta = 2n\pi + \frac{\pi}{2} \text{ or } 2n\pi + 2\alpha \text{ where } \tan^{-1} \frac{3}{7} = \alpha \]

**PART - II**

**Section (A)**

A-3. \( 3 \cos^2 \alpha + \sin^2 \alpha - 2 \cos^2 \alpha + \sin^2 \alpha \)

\[ = 3 \{ 1 - 2 \sin^2 \alpha \} - 2 \{ 1 \times (\cos^2 \alpha + \sin^2 \alpha - \sin^2 \alpha \cos^2 \alpha) \} \]

\[ = 3 - 6 \sin^2 \alpha \cos^2 \alpha - 2 \{ 1 - 3 \sin^2 \alpha \cos^2 \alpha \} \]

\[ = 3 - 6 \sin^2 \alpha \cos^2 \alpha - 2 + 6 \sin^2 \alpha \cos^2 \alpha = 1 \]

A-6. \( \left( 1 + \cos \frac{\pi}{10} \right) \left( 1 + \cos \frac{3\pi}{10} \right) \left( 1 - \cos \frac{3\pi}{10} \right) \left( 1 - \cos \frac{\pi}{10} \right) \)

\[ = \left( 1 - \cos^2 \frac{\pi}{10} \right) \left( 1 - \cos^2 \frac{3\pi}{10} \right) \]

\[ \sin^2 \frac{\pi}{10} \cdot \sin^2 \frac{3\pi}{10} \]

\[ = \left( \frac{\sqrt{5} - 1}{4} \cdot \frac{\sqrt{5} + 1}{4} \right)^2 \]

\[ = \left( \frac{4}{16} \right)^2 \]

\[ = \frac{1}{16} \]

**Section (B)**

B-2. \( 3 \sin \alpha = 5 \sin \beta \)

\[ \Rightarrow \quad \frac{\sin \alpha}{\sin \beta} = \frac{5}{3} \quad \Rightarrow \quad \frac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \frac{8}{2} \quad \Rightarrow \quad \frac{\tan \left( \frac{\alpha + \beta}{2} \right)}{\tan \left( \frac{\alpha - \beta}{2} \right)} = 4 \]

B-7. \( \cot (A + B) = \cot 225^\circ = 1 \quad \Rightarrow \quad \frac{\cot A \cot B - 1}{\cot A + \cot B} = 1 \]

\[ \Rightarrow \quad \cot A \cot B = 1 + \cot A + \cot B \]

Now \( \frac{\cot A \cdot \cot B}{1 + \cot A + \cot B + \cot A \cot B} = \frac{1 + \cot A + \cot B}{2(1 + \cot A + \cot B)} = \frac{1}{2} \)

**Section (C)**

C-3. \( \tan A = \frac{4}{3} \quad \Rightarrow \quad A \rightarrow \text{III}^{\text{a}} \text{ quadrant} \)

\[ 5 \sin 2A + 3 \sin A + 4 \cos A \]

\[ = 10 \sin A \cos A + 3 \sin A + 4 \cos A \]

\[ = 10 \sin A \cos A + 3 \sin A + 4 \cos A \]

\[ = 0 \]

\[ \sin A = -\frac{4}{5} \quad \text{and} \quad \cos A = -\frac{3}{5} \]
C-6. \[ \tan^2 \theta = 2 \tan^2 \phi + 1 \quad \text{... (i)} \]

\[ \cos 2\theta + \sin^2 \phi = \frac{1-\tan^2 \theta}{1+\tan^2 \theta} + \sin^2 \phi \]

\[ = \frac{1-2\tan^2 \phi - 1}{1+2\tan^2 \phi + 1} + \sin^2 \phi = \frac{-2\tan^2 \phi}{2(1+\tan^2 \phi)} + \sin^2 \phi \]

\[ = -\sin^2 \phi + \sin^2 \phi = 0 \]

which is independent of \( \phi \)

C-7*. \[ \sin t + \cos t = \frac{1}{5} \]

\[ \Rightarrow \frac{2\tan \frac{t}{2} + 1 - \tan^2 \frac{t}{2}}{1 + \tan^2 \frac{t}{2}} = \frac{1}{5} \]

\[ \Rightarrow 10 \tan^2 \frac{t}{2} + 5 - 5 \tan^2 \frac{t}{2} = 1 + \tan^2 \frac{t}{2} \]

\[ \Rightarrow 6 \tan^2 \frac{t}{2} - 10 \tan \frac{t}{2} - 4 = 0 \]

\[ \Rightarrow 3 \tan^2 \frac{t}{2} - 6 \tan \frac{t}{2} + \tan \frac{t}{2} - 2 = 0 \]

\[ \Rightarrow 3 \tan \frac{t}{2} \left( \tan \frac{t}{2} - 2 \right) + 1 \left( \tan \frac{t}{2} - 2 \right) = 0 \]

\[ \Rightarrow \tan \frac{t}{2} = 2, \tan \frac{t}{2} = -\frac{1}{3} \]

Section (D) :

D-1. \[ f(\theta) = \sin^4 \theta + \cos^2 \theta \]

\[ = \sin^4 \theta (1 - \cos^2 \theta) + \cos^2 \theta \]

\[ = \sin^4 \theta + \cos^4 \theta - \sin^2 \theta \cos^2 \theta \]

\[ f(\theta) = 1 - \frac{1}{4} \sin^2 2\theta \]

\[ f(\theta)_{\text{max}} = 1 \]

\[ f(\theta)_{\text{min}} = 1 - \frac{1}{4} = 3/4 \]

\[ \therefore \text{Range is } \left[ \frac{3}{4}, 1 \right] \]

D-2*. \[ 1 + 4 \sin \theta + 3 \cos \theta \]

\[ \therefore 4 \sin \theta + 3 \cos \theta \in [-5, 5] \]

\[ \therefore \text{Max. } = 1 + 5 = 6 \]

\[ \text{Min. } = 1 - 5 = -4 \]

Section (E) :

E-2. \[ \text{square & add} \]

\[ a^2 + b^2 = 9 + 16 = 25 \]

E-5*. \[ 1 \text{ radian } \approx 57^\circ \text{ (approx.)} \]

\[ \therefore \sin 1 > \sin 1^\circ \]

\[ \therefore \cos 1^\circ > \cos 1 \]
Section (F):

F-3. \[ A = \tan 6^\circ \tan 42^\circ \]
\[ B = \cot 66^\circ \cot 78^\circ \]

\[ \frac{A}{B} = \tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ \]

\[ \Rightarrow \frac{A}{B} = \frac{\tan 6^\circ \tan (60^\circ - 6^\circ) \tan (60^\circ + 6^\circ)}{\tan 54^\circ} \cdot \tan 78^\circ \tan 42^\circ \]

\[ \Rightarrow \frac{A}{B} = \frac{\tan 18^\circ \cdot \tan (60^\circ - 18^\circ) \tan (60^\circ + 18^\circ)}{\tan 54^\circ} = \frac{\tan 54^\circ}{\tan 54^\circ} \]

\[ \Rightarrow \frac{A}{B} = 1 \quad \Rightarrow \quad A = B \]

F-5*.

\[ \cos \frac{\pi}{10} \cdot \cos \frac{2\pi}{10} \cdot \cos \frac{4\pi}{10} \cdot \cos \frac{8\pi}{10} \cdot \cos \frac{16\pi}{10} = \]

\[ = \frac{\sin^2 \frac{\pi}{10}}{2^5 \sin \frac{\pi}{10}} = \frac{1}{32} \cdot \frac{\sin \frac{3\pi}{10} + \sin \frac{4\pi}{10}}{\sin \frac{\pi}{10}} = \frac{1}{32} \cdot \frac{2 \sin \frac{\pi}{10} \cos \frac{\pi}{10}}{\sin \frac{\pi}{10}} = \frac{1}{16} \cos \frac{\pi}{10} \]

\[ = -\frac{1}{64} \sqrt{10 + 2\sqrt{5}} \]

F-7*.

\[ \cos^2 x + \cos^2 y + \cos^2 z - 2 \cos x \cos y \cos z \]

(Given \( x + y = z \))

\[ = 1 + \cos (x + y) \cos (x - y) + \cos^2 z - 2 \cos x \cos y \cos z \]

\[ = 1 + \cos z \cdot 2 \cos x \cos z - 2 \cos x \cos y \cos z \]

\[ = 1 \]

\[ = \cos (x + y - z) \]

F-9*.

\[ \tan A + \tan B + \tan C = 6, \tan A \tan B = 2 \]

In any \( \triangle ABC, \)

\[ \tan A + \tan B + \tan C = \tan A \tan B \tan C \]

\[ \Rightarrow \quad \tan A + \tan B = 3 \quad \tan C = 3 \]

\[ \therefore \quad \tan A + \tan B + 3 = 6 \]

\[ \Rightarrow \quad \tan A + \tan B = 3 \quad \tan A \tan B = 2 \]

Now \( (\tan A - \tan B)^2 = (\tan A + \tan B)^2 - 4 \tan A \tan B \)

\[ = 9 - 8 = 1 \]

\[ \Rightarrow \quad \tan A - \tan B = \pm 1 \]

\[ \therefore \quad \tan A - \tan B = 1 \quad \text{or} \quad \tan A - \tan B = -1 \]

\[ \tan A + \tan B = 3 \quad \tan A + \tan B = 3 \]

on solving

\[ \tan A = 2 \quad \tan A = 1 \]

\[ \tan B = 1 \quad \tan B = 2 \]

Section (G):

G-3. \[ \tan x + \tan \left( x + \frac{\pi}{3} \right) + \tan \left( x + \frac{2\pi}{3} \right) = 3 \quad \Rightarrow \quad 3 \tan 3x = 3 \]

\[ \Rightarrow \quad \tan 3x = 1 \quad \Rightarrow \quad x = \frac{n\pi}{3} + \frac{\pi}{12}, \quad n \in \mathbb{I} \]
Section (H) :

H-2. \( \sin 7x + \sin 4x + \sin x = 0 \)
\[ \Rightarrow 2 \sin 4x \cos 3x + \sin 4x = 0 \]
\[ \Rightarrow \sin 4x = 0 \quad \text{or} \quad \cos 3x = \pm \frac{2\pi}{9} \]
\[ \Rightarrow 4x = n\pi \quad \text{or} \quad 3x = \frac{2n\pi}{3} \quad \text{or} \quad 3x = \frac{2n\pi}{9} \quad \text{or} \quad \frac{4n\pi}{9} \]

H-4.* \( 2\sin 2x = \sin x + \sin 3x \)
\[ \Rightarrow 2\sin 2x = 2\sin x \cos x \]
\[ \Rightarrow \sin 2x = 0 \quad \text{or} \quad \cos x = \pm 1 \]
\[ \Rightarrow 2x = n\pi \quad \text{or} \quad x = 2m\pi \]
\[ \Rightarrow x = \frac{n\pi}{2}, 2m\pi \]

options (A), (B), (C), (D) are all a part of \( x = \frac{n\pi}{2}. \)

Section (I) :

I-4. \( \cos 2\theta + 3 \cos \theta = 0 \)
\[ \Rightarrow 2 \cos^2 \theta + 3 \cos \theta - 1 = 0 \]
\[ \Rightarrow \cos \theta = \frac{3 \pm \sqrt{9 + 8}}{4} = \frac{-3 \pm \sqrt{17}}{4} \]

As \(-1 \leq \cos \theta \leq 1\) \quad \therefore \quad \cos \theta = \frac{-3 + \sqrt{17}}{4} \quad \text{only} \quad \Rightarrow \theta = 2n\pi \pm \alpha \quad \text{where} \quad \cos \alpha = \frac{\sqrt{17} - 3}{4} \]

I-5. \( \sin \theta + 7 \cos \theta = 5 \)
\[ \Rightarrow \frac{2t}{1 + t^2} + \frac{7(1-t^2)}{1 + t^2} = 5 \]
\[ \Rightarrow 2t + 7 - 7t^2 = 5 + 5t^2 \]
\[ \Rightarrow 2t + 7 - 7t^2 = 5 + 5t^2 \]
\[ \Rightarrow \tan \frac{\theta}{2} \text{ is root of } 12t^2 - 2t - 2 = 0 \quad \text{or} \quad 6t^2 - t - 1 = 0. \]

Section (J) :

J-1. \( \tan \theta = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{4}, \frac{7\pi}{4} \quad \text{in} \quad [0, 2\pi] \)
\[ \cos \theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{7\pi}{4} \quad \text{in} \quad [0, 2\pi] \quad \therefore \quad \text{common value is} \quad x = \frac{7\pi}{4} \]
\[ \therefore \quad \text{general solution is} \quad 2n\pi + \frac{7\pi}{4}, \quad n \in \mathbb{I}. \]

J-3.* \( E = \sin x - \cos^2 x - 1 \)
\[ \Rightarrow \quad E = \sin x - 1 + \sin^2 x - 1 = \sin^2 x + \sin x - 2 \]
\[ = \left( \sin x + \frac{1}{2} \right)^2 - \frac{9}{4} \]
\[ \text{assumes least value} \]
\[ \text{when} \quad \sin x = -\frac{1}{2} \quad \Rightarrow \quad x = n\pi + (-1)^n \left( -\frac{\pi}{6} \right). \]

**EXERCISE # 2**

**PART - I**

3. \( \sin x + \sin y = a \quad \text{......(1)} \)
\( \cos x + \cos y = b \quad \text{......(2)} \)
\[ \frac{2\sin \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right)}{2\cos \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right)} = \frac{a}{b} \]
\[ \tan \left( \frac{x + y}{2} \right) = \frac{a}{b} \]

\[ \Rightarrow \quad \sin \left( \frac{x + y}{2} \right) = \frac{a}{\sqrt{a^2 + b^2}}, \quad \cos \left( \frac{x + y}{2} \right) = \frac{b}{\sqrt{a^2 + b^2}} \]

\[ \Rightarrow \quad \sin (x + y) = 2 \sin \left( \frac{x + y}{2} \right) \cos \left( \frac{x + y}{2} \right) = \frac{2ab}{a^2 + b^2} \]

Now for \( \tan \left( \frac{x - y}{2} \right) \)

\[ \Rightarrow \quad (1)^2 + (2)^2 \]

\[ 1 + 1 + 2 \cos (x - y) = a^2 + b^2 \]

\[ \cos (x - y) = \frac{a^2 + b^2 - 2}{2} \]

\[ \therefore \quad \tan^2 \left( \frac{x - y}{2} \right) = \frac{1 - \cos(x - y)}{1 + \cos(x - y)} \quad \Rightarrow \quad \tan^2 \left( \frac{x - y}{2} \right) = \frac{1 - \left( \frac{a^2 + b^2 - 2}{2} \right)}{1 + \frac{a^2 + b^2 - 2}{2}} \]

\[ \Rightarrow \quad \tan \left( \frac{x - y}{2} \right) = \pm \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}} \]

6. \( \tan \alpha = \frac{p}{q} \)

\[ \text{LHS} = \frac{1}{2} (p \csc 2\beta - q \sec 2\beta) \times \frac{\sqrt{p^2 + q^2}}{\sqrt{p^2 + q^2}} \]

\[ = \frac{1}{2} \left( \frac{p}{\sqrt{p^2 \times q^2}} \csc 2\beta - \frac{q}{\sqrt{p^2 \times q^2}} \sec 2\beta \right) \times \sqrt{p^2 + q^2} \]

\[ = \frac{1}{2} \left( \frac{\sin \alpha \cos 2\beta - \cos \alpha \sin 2\beta}{\sin 2\beta \cos 2\beta} \right) \times \sqrt{p^2 + q^2} \]

\[ = \frac{\sin(\alpha - 2\beta)}{\sin 4\beta} \times \sqrt{p^2 + q^2} = \frac{\sin 4\beta}{\sin 4\beta} \times \sqrt{p^2 + q^2} \quad (\because \alpha = 6\beta) \]

7. (i) \( \cot \frac{71^0}{2} = \tan \frac{82^0}{2} = \frac{\cos \frac{71^0}{2}}{\sin \frac{71^0}{2}} = \frac{2 \cos^2 \frac{71^0}{2}}{\sin 15^0} \)

\[ = \frac{1 + \cos(45^0 - 30^0)}{\sin(45^0 - 30^0)} = \frac{1 + \frac{\sqrt{3} + 1}{2\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}} = \frac{2\sqrt{2} + \sqrt{3} + 1}{3 - 1} \]

\[ = \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{3 - 1} = \sqrt{2} + \sqrt{3} + 2 + \sqrt{6} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6} \]
$= (\sqrt{2} + \sqrt{3}) (\sqrt{2} + 1)$

(ii) \[\tan \frac{142^\circ}{2} = - \cot \frac{52^\circ}{2} \quad \frac{-1}{\tan \frac{52^\circ}{2}} = \frac{-1}{\tan \left(\frac{45 + 7^\circ}{2}\right)}\]

\[\frac{1 - \tan \frac{7^\circ}{2}}{1 + \tan \frac{7^\circ}{2}} = - \frac{\cos \frac{7^\circ}{2} - \sin \frac{7^\circ}{2}}{\cos \frac{7^\circ}{2} + \sin \frac{7^\circ}{2}}\]

\[= \frac{\left(\cos \frac{7^\circ}{2} - \sin \frac{7^\circ}{2}\right)^2}{\cos 15^\circ} = - \frac{1 - \sin 15^\circ}{\cos 15^\circ} = - \frac{1 - \sqrt{3} - \frac{1}{2\sqrt{2}}}{\sqrt{3} + \frac{1}{2\sqrt{2}}} = - \frac{(2\sqrt{2} - \sqrt{3} + 1)(\sqrt{3} - 1)}{2}\]

\[= - \frac{\left[2\sqrt{2} (\sqrt{3} - 1) - (\sqrt{3} - 1)^2\right]}{2} = - \frac{\left[2\sqrt{2} (\sqrt{3} - 1) - (4 - 2\sqrt{3})\right]}{2}\]

\[= - \left[\sqrt{2}(\sqrt{3} - 1) - (2 - \sqrt{3})\right] = - \sqrt{6} + \sqrt{2} + 2 - \sqrt{3} = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}\]

9. (i) \[\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ\]

\[= (\tan 9^\circ + \tan 81^\circ) - (\tan 27^\circ + \tan 63^\circ)\]

\[= \frac{\sin 90^\circ}{\cos 9^\circ \cos 81^\circ} - \frac{\sin 90^\circ}{\cos 27^\circ \cos 63^\circ} = \frac{2}{2 \sin 9^\circ \cos 9^\circ} - \frac{2}{2 \sin 27^\circ \cos 27^\circ}\]

\[= \frac{2}{\sin 18^\circ} - \frac{2}{\sin 54^\circ} = \frac{1}{\sqrt{5} - 1} - \frac{1}{\sqrt{5} + 1}\]

\[= \frac{8(\sqrt{5} + 1 - \sqrt{5} + 1)}{4} = 4\]

(ii) \[\cosec 10^\circ - \sqrt{3} \sec 10^\circ = 2 \left(\frac{1}{2} \cos 10^\circ - \sqrt{3} \sin 10^\circ\right) \times \frac{1}{\sin 10^\circ \cos 10^\circ} = \frac{2}{2} = 4\]

(iii) \[2\sqrt{2} \sin 10^\circ \left(\frac{\sec 5^\circ + \cos 40^\circ}{2} - \sin 35^\circ\right)\]

\[= \frac{2\sqrt{2}}{2} \left(\frac{2 \sin 5^\circ \cos 5^\circ \sec 5^\circ}{2} + \frac{2 \sin 5^\circ \cos 5^\circ \cos 40^\circ}{\sin 5^\circ} - \frac{2 \sin 35^\circ \sin 10^\circ}{2}\right)\]

\[= 2\sqrt{2} (\sin 5^\circ + 2\cos 45^\circ + \cos 35^\circ - \cos 25^\circ + \cos 45^\circ)\]

\[= 2\sqrt{2} (\sin 5^\circ + 2\cos 45^\circ + 2\sin 30^\circ \sin (-5^\circ))\]

\[= 2\sqrt{2} \sqrt{2} = 4\]

(iv) \[\cot 70^\circ + 4 \cos 70^\circ = \frac{\cos 70^\circ + 4 \cos 70^\circ}{\sin 70^\circ} = \frac{\cos 70^\circ + 4 \cos 70^\circ \sin 70^\circ}{\sin 70^\circ}\]

\[= \frac{\cos 70^\circ + 2 \sin 140^\circ}{\sin 70^\circ}\]

\[= \frac{(\cos 70^\circ + \sin 140^\circ) + \sin 140^\circ}{\sin 70^\circ} = \frac{(\sin 20^\circ + \sin 140^\circ) + \sin 140^\circ}{\sin 70^\circ}\]
\[
\begin{align*}
\tan 10^\circ - \tan 50^\circ + \tan 70^\circ &= \tan 10^\circ - \frac{\sqrt{3} + \tan 10^\circ}{1 - \sqrt{3} \tan 10^\circ} + \frac{\sqrt{3} - \tan 10^\circ}{1 + \sqrt{3} \tan 10^\circ} \\
&= \frac{9 \tan 10^\circ - 3 \tan^3 10^\circ}{1 - 3 \tan^2 10^\circ} \\
&= 3 \left( \frac{3 \tan 10^\circ - \tan^3 10^\circ}{1 - 3 \tan^2 10^\circ} \right) = 3 \tan 30^\circ = \sqrt{3}
\end{align*}
\]

11. \[\begin{align*}
\frac{1}{A_1A_2} &= \frac{1}{A_1A_3} + \frac{1}{A_1A_4} \\
\therefore \quad OA_1 &= OA_2 = OA_3 = OA_4 = r \text{ (say)} \\
\angle A_1OA_2 &= \frac{2\pi}{n}, \quad \angle A_1OA_3 = \frac{4\pi}{n}, \quad \angle A_1OA_4 = \frac{6\pi}{n}
\end{align*}\]

\[\begin{align*}
\Rightarrow \quad \sin \frac{\pi}{n} &= \sin \frac{2\pi}{n} + \sin \frac{3\pi}{n} \\
\Rightarrow \quad \sin \frac{2\pi}{n} \left[ \sin \frac{3\pi}{n} - \sin \frac{\pi}{n} \right] &= \sin \frac{3\pi}{n} \sin \frac{\pi}{n} \\
\Rightarrow \quad 2 \sin \frac{2\pi}{n} \cos \frac{2\pi}{n} &= \sin \frac{3\pi}{n} \\
\Rightarrow \quad \sin \frac{4\pi}{n} &= \sin \frac{3\pi}{n} \Rightarrow \quad \frac{4\pi}{n} = \pi - \frac{3\pi}{n} \\
\Rightarrow \quad 4\pi &= n\pi - 3\pi \quad \Rightarrow \quad n = 7
\end{align*}\]

13. \[\begin{align*}
P_n - P_{n-2} &= \cos^n \theta + \sin^n \theta - \cos^{n-2} \theta - \sin^{n-2} \theta \\
&= \cos^{n-2} \theta (\cos^2 \theta - 1) + \sin^{n-2} \theta (\sin^2 \theta - 1) \\
&= \cos^{n-2} \theta (-\sin^2 \theta) + \sin^{n-2} \theta (-\cos^2 \theta) \\
&= (-\sin^2 \theta \cos^2 \theta) P_{n-4}
\end{align*}\]

\[\begin{align*}
\therefore \quad n &= 4 \\
P_4 &= (-\sin^2 \theta \cos^2 \theta) P_0 \\
P_4 &= P_2 - 2 \sin^2 \theta \cos^2 \theta \\
&= 1 - 2 \sin^2 \theta \cos^2 \theta
\end{align*}\]

Similarly, we can prove the other result also.

15. \[\begin{align*}
&\tan^2 \alpha + 2 \tan \alpha \cdot \tan 2\beta = \tan^2 \beta + 2 \tan \beta \cdot \tan 2\alpha \\
\Rightarrow \quad &\left( \tan^2 \alpha - \tan^2 \beta \right) + 4 \tan \alpha \tan \beta \left( \frac{1}{1 - \tan^2 \beta} - \frac{1}{1 - \tan^2 \alpha} \right) = 0 \\
\Rightarrow \quad &\left( \tan^2 \alpha - \tan^2 \beta \right) + 4 \tan \alpha \tan \beta \left( \frac{\tan^2 \beta - \tan^2 \alpha}{1 - \tan^2 \alpha}(1 - \tan^2 \beta) \right) = 0
\end{align*}\]
\[ (\tan^2 \alpha - \tan^2 \beta) \left( \frac{1 - 4 \tan \alpha \tan \beta}{(1 - \tan^2 \alpha)(1 - \tan^2 \beta)} \right) = 0 \]
\[ \Rightarrow (\tan^2 \alpha - \tan^2 \beta) \cdot (1 - \tan 2\alpha \cdot \tan 2\beta) = 0 \]
\[ \Rightarrow \tan^2 \alpha = \tan^2 \beta \quad \text{or} \quad \tan 2\alpha \cdot \tan 2\beta = 1 \]

L.H.S. = \tan^2 \alpha + 2 \tan \alpha \cdot \frac{1}{\tan 2\alpha}

= \tan^2 \alpha + \frac{2 \tan \alpha}{2 \tan \alpha} \cdot (1 - \tan^2 \alpha) = 1

R.H.S. = \tan^2 \beta + 2 \tan \beta \cdot \frac{1}{\tan 2\beta}

= \tan^2 \beta + \frac{2 \tan \beta}{2 \tan \beta} \cdot (1 - \tan^2 \beta) = 1

19. \[ \sqrt{13 - 18 \tan x} = 6 \tan x - 3 \quad \text{..................................(1)} \]
\[ \Rightarrow 13 - 18 \tan x = 36 \tan^2 x + 9 - 36 \tan x \]
\[ \Rightarrow \tan x = \frac{2}{3} \quad \frac{1}{6} \]

Put in (1) \[ \Rightarrow \text{tan } x = \frac{2}{3} \text{ is correct} \]
\[ \Rightarrow x = n\pi + \tan^{-1} \left( \frac{2}{3} \right) \]
\[ = n\pi + \alpha, \pi + \alpha, -\pi + \alpha, -2\pi + \alpha \quad \text{in \((-2\pi, 2\pi))} \]

22. \[ \tan \theta + \sin \phi = \frac{3}{2} \quad \text{.....(1)} \]

As \[ \tan^2 \theta + \cos^2 \phi = \frac{7}{4} \]
\[ \Rightarrow \left( \frac{3}{2} - \sin \phi \right)^2 + \cos^2 \phi = \frac{7}{4} \Rightarrow \left( \frac{9}{4} + \sin^2 \phi - 3 \sin \phi + \cos^2 \phi \right) = \frac{7}{4} \]
\[ \Rightarrow \frac{9}{4} = 3 \sin \phi \Rightarrow \sin \phi = \frac{1}{2} \Rightarrow \phi = n\pi + (-1)^n \frac{\pi}{6} \]

from (1), \[ \tan \theta = \frac{3}{2} - \sin \phi = \frac{3}{2} - \frac{1}{2} = 1 \Rightarrow \theta = n\pi + \frac{\pi}{4} \]

23. \[ a \cos 2\theta + b \sin 2\theta = c \]
\[ \Rightarrow \frac{a(1-t^2)}{1+t^2} + \frac{b(2t)}{1+t^2} = c \quad \text{where } t = \tan \theta \]
\[ \Rightarrow (c + a)t^2 - 2bt + (c - a) = 0 \Rightarrow t_1 + t_2 = \frac{2b}{c+a}, t_1 t_2 = \frac{c-a}{c+a} \]
\[ \therefore \cos^2 \alpha + \cos^2 \beta = \frac{1 + \cos 2\alpha + 1 + \cos 2\beta}{2} = 1 + \frac{1}{2} [\cos 2\alpha + \cos 2\beta] \]
\[ = 1 + \frac{1}{2} \left[ \frac{1-t_1^2}{1+t_1^2} + \frac{1-t_2^2}{1+t_2^2} \right] \]

simplifying and using values for \( t_1, t_2 \) we get
\[ \cos^2 \alpha + \cos^2 \beta = 1 + \frac{ac}{a^2 + b^2} = \frac{a^2 + b^2 + ac}{a^2 + b^2} \].
27. \[ \text{RHS} = 3x^2 + 2x + 3 \]

Minimum value \[ \frac{4(3)(3) - 4}{4(3)} = \frac{32}{12} = \frac{8}{3} > 2 \]

whereas LHS \( \leq 2 \) \quad \therefore \quad \text{no solution.}

\[ \text{PART - II} \]

2. For dodecagon

\[ \angle \text{A'OB'} = \frac{2\pi}{12} = 30^\circ \]

\[ \Rightarrow \quad \angle \text{OA'B'} = \angle \text{OB'A} = 75^\circ \]

\[ \Rightarrow \quad \frac{R}{\sin 75^\circ} = \frac{\sqrt{3} - 1}{\sin 30^\circ} \]

\[ \Rightarrow \quad R = \frac{(\sqrt{3} - 1)(\sqrt{3} + 1)}{2\sqrt{2} \times \frac{1}{2}} \quad \Rightarrow \quad R = \sqrt{2} \]

For hexagon \( \angle \text{AOB} = \frac{2\pi}{6} = 60^\circ \)

\[ \Rightarrow \quad \triangle \text{AOB is equilateral} \quad \Rightarrow \quad \text{AB} = R = \sqrt{2} \]

6. \( A + B + C = \pi \)

\[ \frac{\sin \left( \frac{A + C}{2} \right)}{\sin \frac{C}{2}} = \frac{k}{1} \]

\[ \frac{\sin \left( \frac{A + C}{2} \right) - \sin \frac{C}{2}}{\sin \left( \frac{A + C}{2} \right) + \sin \frac{C}{2}} = \frac{k - 1}{k + 1} \]

\[ \Rightarrow \quad \frac{2 \cos \frac{A + C}{2} \sin \frac{A}{2}}{2 \sin \left( \frac{A + C}{2} \right) \cos \frac{A}{2}} = \frac{k - 1}{k + 1} \quad \Rightarrow \quad \tan \frac{B}{2} \tan \frac{A}{2} = \frac{k - 1}{k + 1} \]

8. \[ 4 \cos^2 \left( \frac{\pi}{4} - \frac{x}{2} \right) + \sqrt{4 \sin^4 x + 4 \sin^2 x \cos^2 x} \]

\[ = 4 \cos^2 \left( \frac{\pi}{4} - \frac{x}{2} \right) + 2 \sin x = 4 \cos^2 \left( \frac{\pi}{4} - \frac{x}{2} \right) - 2 \sin x \]

\[ = 2 \left( 1 + \cos \left( \frac{\pi}{2} - x \right) \right) - 2 \sin x = 2 \]

10. \[ \sum \cos A \cos ec B \cos ec C = \frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin A \sin C} + \frac{\cos C}{\sin A \sin B} \]

\[ = \frac{\cos A \sin A + \cos B \sin B + \cos C \sin C}{\sin A \sin B \sin C} \]
13. \[ \frac{\cos 6x + \cos 4x}{\cos 5x + \cos 3x + 10 \cos x} \]

\[ = \frac{2 \cos 5x \cos x + 5 \times 2 \cos 3x \cos x + 10 \times 2 \cos^2 x}{\cos 5x + \cos 3x + 10 \cos x} \]

\[ = 2 \cos x \left[ \frac{\cos 5x + 5 \cos 3x + 10 \cos x}{\cos 5x + \cos 3x + 10 \cos x} \right] \]

\[ = 2 \cos x \]

15. \[ \frac{\cos 3\theta}{\cos \theta} = 4 \cos^2 \theta - 3 = 2 (1 + \cos 2\theta) - 3 \]

\[ = 2 \cos 2\theta - 1 = 2 \cos (\alpha - \beta) - 1 \]

\[ (\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) + 2 \cos (\alpha - \beta) = a^2 + b^2 \]

\[ 2 \cos (\alpha - \beta) = a^2 + b^2 - 2 \quad \Rightarrow \quad \frac{\cos 3\theta}{\cos \theta} = a^2 + b^2 - 3 \]

20. \[ \sin 3\theta = 4 \sin \theta \sin 2\theta \sin 4\theta \]

\[ \Rightarrow \sin 3\theta = (2 \sin \theta) (2 \sin 2\theta \sin 4\theta) \quad \Rightarrow \quad 3 \sin \theta - 4 \sin^3 \theta = 2 \sin \theta (\cos 2\theta - \cos 6\theta) \]

\[ \Rightarrow 3 - 4 \sin^2 \theta = 2(\cos 2\theta - \cos 6\theta) \quad \text{or} \quad \sin \theta = 0 \]

\[ \Rightarrow 3 - 2(1 - \cos 2\theta) = 2 \cos 2\theta - 2 \cos 6\theta \quad \text{or} \quad \sin \theta = 0 \]

\[ \Rightarrow 1 = -2 \cos 6\theta \quad \Rightarrow \cos 6\theta = \frac{-1}{2} \quad \text{or} \quad \sin \theta = 0 \]

\[ \therefore \sin \theta = 0 \text{ or } \cos 6\theta = \frac{-1}{2} \]

\[ \Rightarrow \theta = n\pi \quad \text{or} \quad \theta = \frac{2n\pi \pm \frac{2\pi}{3}}{6} = \frac{n\pi}{3} \pm \frac{\pi}{9} \]

So eight solutions.

22. \[ 2 \cos x = \sqrt{2} + 2 \sin 2x \]

\[ \Rightarrow \sqrt{2} \cos x = \sqrt{1 + \sin^2 2x} = |\sin x + \cos x| \quad \Rightarrow \quad \cos x = \frac{1}{\sqrt{2}} (\sin x + \cos x) \]

\[ \Rightarrow \cos x = \left| \sin \left( x + \frac{\pi}{4} \right) \right| \]

\[ \Rightarrow \text{see from graph or we can put values given in options to verify.} \]

25. \[ 2 \tan^2 x - 5 \sec x - 1 = 0 \]

\[ \Rightarrow 2(\sec^2 x - 1) - 5 \sec x - 1 = 0 \]

\[ \Rightarrow 2 \sec^2 x - 5 \sec x - 3 = 0 \]

\[ \Rightarrow \sec x = \frac{6}{2}, -1 = 3, \frac{-1}{2} \]
\[ \Rightarrow \sec x = 3 \quad \left( \sec x = -\frac{1}{2} \right) \]

\[ y = \frac{1}{3} \]

\[ \Rightarrow \cos x = \frac{1}{3} \quad \Rightarrow \text{7 solutions in } \left[ 0, \frac{15\pi}{2} \right] \quad \therefore n = 15. \]

28. \[4 \cos^3 x - 4 \cos^2 x + \cos x - 1 = 0\]

\[ (4 \cos^2 x + 1)(\cos x - 1) = 0\]

\[ \Rightarrow \cos x = 1\]

solutions in the interval \([0, 3\pi]\) are \(0, 2\pi, 4\pi, \ldots, 100\pi\)

\[ \therefore \text{arithmetic mean } = \frac{0 + 2\pi + 4\pi + \ldots + 100\pi}{51} = \frac{50\pi}{1} \]

29. \[ h = \sqrt{(\cos 2\alpha + \cos 2\beta + 2\cos(\alpha + \beta))^2 + (\sin 2\alpha + \sin 2\beta + 2\sin(\alpha + \beta))^2} \]

\[ \Rightarrow h = [4 \cos^2(\alpha + \beta) (\cos (\alpha - \beta) + 1)^2 + 4 \sin^2(\alpha + \beta) (\cos (\alpha - \beta) + 1)^2]^{1/2} \]

\[ \Rightarrow h = [4(\cos (\alpha - \beta) + 1)^2 (\cos^2(\alpha + \beta) + \sin^2(\alpha + \beta))]^{1/2} \]

\[ \Rightarrow h = 2 (1 + \cos (\alpha - \beta)) \quad \Rightarrow \quad h = 2 \times 2 \cos^2 \left( \frac{\alpha - \beta}{2} \right) \]

\[ \Rightarrow h = 4 \cos^2 \left( \frac{\alpha - \beta}{2} \right) \]

32. \[ y = a \cos^2 x + 2b \sin x \cos x + c \sin^2 x \quad & \quad \tan x = \frac{2b}{a - c} \]

\[ z = a \sin^2 x - 2b \sin x \cos x + c \cos^2 x \]

\[ \Rightarrow y + z = a + c \]

and \[ y - z = (a - c)(\cos^2 x - \sin^2 x) + 4b \sin x \cos x \]

\[ = (a - c) \cos 2x + 2b \sin 2x \quad (\because 2b = (a - c) \tan x) \]

\[ = (a - c) [\cos 2x + \tan x . \sin 2x] = (a - c) \left[ 2 \cos 2x + \frac{\sin x}{\cos x} \right] \sin 2x \]

\[ = \frac{(a - c) \cos(2x - x)}{\cos x} = (a - c). \]

33. \[ \frac{2 \cos \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right)}{2 \cos \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right)} + \frac{2 \sin \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right)}{-2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right)} \]

\[ = \cot^n \left( \frac{A - B}{2} \right) + (-1)^n \cot^n \left( \frac{A - B}{2} \right) \]

\[ \begin{cases} 0 : n \in \text{odd} \\ 2 \cot^n \left( \frac{A - B}{2} \right) : n \in \text{even} \end{cases} \]
34. \[ \sin^6 x + \cos^6 x = a^2 \]
\[ \Rightarrow (\sin^2 x + \cos^2 x) (\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x) = a^2 \]
\[ \Rightarrow (\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x = a^2 \]
\[ \Rightarrow 1 - \frac{3}{4} \sin^2 2x = a^2 \]
\[ \Rightarrow 0 \leq \frac{4}{3} (1 - a^2) \leq 1 \]
\[ 1 - a^2 \geq 0 \quad \text{and} \quad 4 - 4a^2 \leq 3 \]
\[ a^2 \leq 1 \quad \text{and} \quad \frac{1}{4} \leq a^2 \]
\[ -1 \leq a \leq 1 \quad \text{and} \quad a \geq \frac{1}{2} \text{ or } a \leq -\frac{1}{2} \]
\[ a \in \left[ -1, -\frac{1}{2} \right] \cup \left[ \frac{1}{2}, 1 \right] \]

38. \[ \cos 15x = \sin 5x \]
\[ \cos 15x = \cos (\frac{\pi}{2} - 5x) \quad \text{or} \quad \cos \left( \frac{3\pi}{2} + 5x \right) \]
\[ 15x = 2n\pi \pm \left( \frac{\pi}{2} - 5x \right) \quad \text{or} \quad 15x = 2n\pi \pm \left( \frac{3\pi}{2} + 5x \right) \]
\[ \Rightarrow x = \frac{n\pi}{10} + \frac{\pi}{40}, n \in I \quad \text{or} \quad x = \frac{n\pi}{5} + \frac{3\pi}{20}, n \in I \]
\[ \text{and} \quad x = \frac{n\pi}{5} - \frac{\pi}{20}, n \in I \quad \text{and} \quad x = \frac{n\pi}{10} - \frac{3\pi}{40}, n \in I \]

40. \[ \sin^2 x + 2 \sin x \cos x - 3 \cos^2 x = 0 \]
\[ \text{case-I : } \cos x \neq 0 \quad \therefore \quad \tan^2 x + 2 \tan x - 3 = 0 \]
\[ \Rightarrow \tan x = 3, 1 \quad \Rightarrow x = n\pi + \tan^{-1}(-3), n\pi + \frac{\pi}{4} \]
\[ \text{case-II : } \cos x = 0 \quad \Rightarrow 1 + 0 - 0 = 0 \text{ not true.} \]

EXERCISE # 3

3. (A) \[ \sin^2 \theta + 3 \cos \theta = 3 \]
\[ \Rightarrow 1 - \cos^2 \theta + 3 \cos \theta = 3 \]
\[ \Rightarrow \cos^2 \theta - 3 \cos \theta + 2 = 0 \]
\[ \Rightarrow \cos \theta = 1 \quad (\because \cos \theta \neq 2) \]
\[ \Rightarrow \theta = 0 \text{ in } [-\pi, \pi] \]
\[ \therefore \quad \text{No. of solution} = 1 \]

(B) \[ \sin x \cdot \tan 4x = \cos x \]
\[ \Rightarrow \sin x \cdot \frac{\sin 4x}{\cos 4x} = \cos x \]
\[ \Rightarrow \sin x \sin 4x - \cos x \cos 4x = 0 \]
\[ \Rightarrow \cos 5x = 0 \]
\[ \Rightarrow 5x = (2n + 1) \frac{\pi}{2} \]
\[ \Rightarrow x = (2n + 1) \frac{\pi}{10} \]
\[ \Rightarrow x = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10} \text{ in } (0, \pi) \]
So there are five solutions.

(C) \[ (1 - \tan^2 \theta) \sec^2 \theta + 2 \tan^2 \theta = 0 \]
\[ \Rightarrow (1 - \tan^4 \theta) + 2 \tan^2 \theta = 0 \]
\[ \Rightarrow (1 - x^2) + 2x = 0 \quad \text{where } x = \tan^2 \theta \]
\[2^x = x^2 - 1 \Rightarrow x = 3\]

\[\tan^2 \theta = 3 \Rightarrow \tan \theta = \pm \sqrt{3} \]

\[\therefore \text{ Number of solutions } = 2\]

(D) \[\lceil \sin x \rceil + \lfloor \sqrt{2} \cos x \rfloor = -3\]

\[\Rightarrow [\sin x] = -1 \text{ and } [\sqrt{2} \cos x] = -2\]

\[\Rightarrow -2 \leq \cos x < -\frac{1}{\sqrt{2}} \]

\[\Rightarrow 2 \pi < 2x < \frac{5\pi}{2}\]

\[\Rightarrow [\sin 2x] = 0\]

4. (A)

(B) \[\sin x = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}\]

\[\Rightarrow \sin x = 1 - \sqrt{2}\]

As \(\sin x\) takes at least four values in \([0, \pi]\)

\[\therefore n \geq 4\]

(C) \[1 + \sin^4 x = \cos^2 3x\]

L.H.S. \(\geq 1\) and R.H.S. \(\leq 1\)

\[\Rightarrow \text{L.H.S.} = \text{R.H.S.} = 1 \Rightarrow \sin^4 x = 0 \text{ and } \cos^2 3x = 1\]

\[\Rightarrow x = n\pi \text{ and } 3x = m\pi\]

\[\Rightarrow x = n\pi \text{ and } x = \frac{m\pi}{3}\]

\[\Rightarrow x = n\pi\]

\[\Rightarrow x = -2\pi, -\pi, 0, \pi, 2\pi \text{ in } \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]\]

\[\therefore \text{ Number of solutions } = 5\]

(D) A, B, C are in A.P. \(\Rightarrow B = 60^\circ\)

As \(\sin (2A + B) = \frac{1}{2}\) \(\Rightarrow 2A + B = 30^\circ \text{ or } 150^\circ\)

\[\Rightarrow 2A = -30^\circ \text{ or } 90^\circ \Rightarrow 2A = 90^\circ \Rightarrow A = 45^\circ\]

\[\Rightarrow C = 180^\circ - A - B = 75^\circ = \frac{5\pi}{12} \therefore p = 12.\]

Comprehension #1 (5, 6, 7)

5. Given \(\cos \alpha + \cos \beta = a\) \(\Rightarrow 2 \cos \left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right) = a\) .... (i)

and \(\sin \alpha + \sin \beta = b\)
\[ 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) = b \text{ .... (ii)} \]

by (i) & (ii)

\[ \tan \left( \frac{\alpha + \beta}{2} \right) = \frac{b}{a} \Rightarrow \tan \theta = \frac{b}{a} \]

\[ \therefore \quad \sin 2\theta + \cos 2\theta \]

\[ \frac{2 \left( \frac{b}{a} \right)}{1 + \frac{b^2}{a^2}} = \frac{1 - \frac{b^2}{a^2}}{1 + \frac{b^2}{a^2}} \]

\[ = \frac{2ab}{a^2 + b^2} \]

\[ = \frac{a^2 + b^2 - 2b^2 + 2ab}{a^2 + b^2} \]

\[ = \frac{a^2 + 2ab}{a^2 + b^2} \]

\[ = 1 + \frac{2b(a - b)}{a^2 + b^2} \]

\[ \therefore \quad n = 2 \]

5. \( \sin^n A = x \)

\[ \Rightarrow \quad \sin^2 A = x \]

\[ \therefore \quad \sin A \cos 2A \sin 3A \cos 4A = \sin A (2 \sin A \cos A) (3 \sin A - 4 \sin^3 A) (4 \sin A \cos A (1 - 2 \sin^2 A)) \]

\[ = 8 \sin^4 A (1 - \sin^2 A) (1 - 2 \sin^2 A) (3 - 4 \sin^2 A) \]

If we put \( \sin^2 A = x \), then given expression is a polynomial of degree 5 in x.

7. \( p = 5 \)

\[ \therefore \quad \sin x + (p - 5), \cos x, \tan x \]

\[ \Rightarrow \quad \sin x, \cos x, \tan x \text{ are in G.P.} \]

\[ \therefore \quad \cos^2 x = \sin x \tan x \]

\[ \therefore \quad \cos^3 x = \sin^2 x \]

\[ \therefore \quad \cos^3 x = 1 - \cos^2 x \]

\[ \therefore \quad \cos^3 x + \cos^2 x = 1 \]

taking cube both sides

\[ \therefore \quad \cos^6 x = \cos^6 x + 3 \cos^4 x - 1 = 0 \]

Comprehension # 4

14. \( \sin^6 x + \cos^6 x < \frac{7}{16} \Rightarrow 1 - 3\sin^2 x \cos^2 x < \frac{7}{16} \)

\[ \Rightarrow \quad \sin^2 x \cos^2 x > \frac{3}{16} \Rightarrow \sin^2 2x > \frac{3}{4} \]

\[ \Rightarrow \quad 1 - \cos 4x > \frac{3}{4} \Rightarrow 1 - \cos 4x > \frac{3}{2} \]

\[ \Rightarrow \quad \cos 4x < -\frac{1}{2} \Rightarrow \text{Principal is value} \quad 4x \in \left( \frac{2\pi}{3}, \frac{4\pi}{3} \right) \]

\[ \Rightarrow \quad \text{General value is} \]

\[ 4x \in \left( 2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{4\pi}{3} \right) \]

\[ \Rightarrow \quad x \in \left( \frac{n\pi}{2} + \frac{\pi}{6}, \frac{n\pi}{2} + \frac{\pi}{3} \right), n \in \mathbb{I} \]
15. \[ \cos 2x + 5 \cos x + 3 \geq 0 \]
\[ \Rightarrow 2 \cos^2 x + 5 \cos x + 2 \geq 0 \]
\[ \Rightarrow (\cos x + 2)(2 \cos x + 1) \geq 0 \]
\[ \Rightarrow \cos x \geq -\frac{1}{2} \]
\[ \Rightarrow x \in \left[ -\frac{2\pi}{3}, \frac{2\pi}{3} \right] \]

16. \[ 2 \sin^2 \left( x + \frac{\pi}{4} \right) + \sqrt{3} \cos 2x \geq 0 \]
\[ \Rightarrow 1 - \cos \left( 2x + \frac{\pi}{2} \right) + \sqrt{3} \cos 2x \geq 0 \]
\[ \Rightarrow \sqrt{3} \cos 2x + \sin 2x \geq -1 \]
\[ \Rightarrow \sin \left( 2x + \frac{\pi}{3} \right) \geq -\frac{1}{2} \]
\[ \Rightarrow 2x + \frac{\pi}{3} \in \left[ 2n\pi - \frac{\pi}{6}, 2n\pi + \frac{7\pi}{6} \right] \]
\[ \Rightarrow x \in \left[ \frac{3\pi - 2\pi}{4}, \frac{3\pi + 5\pi}{12} \right] \]
\[ \Rightarrow x \in \left[ -\frac{\pi}{4}, -\frac{7\pi}{12} \right] \cup \left[ -\frac{5\pi}{12}, \frac{3\pi}{4} \right] \in [-\pi, \pi] \]

19. Statement-1 :  \[ \sum \cos A \cosec B \cosec C \]
\[ = \sum \frac{\cos A}{\sin B \sin C} = \sum \frac{\sin A \cos A}{\sin A \sin B \sin C} = \frac{\sum \sin 2A}{2 \sin A \sin B \sin C} \]
\[ = \frac{4 \sin A \sin B \sin C}{2 \sin A \sin B \sin C} = 2 \]
Statement-2 :  \[ \sum \tan A \tan B = 1 \] iff the triangle ABC is right angled  \[ \therefore \] Statement is false

20. \[ y = \frac{\tan \theta}{\tan 3\theta} = \frac{\tan \theta}{3 \tan \theta - \tan^3 \theta} \]
\[ \Rightarrow y = \frac{1 - 3 \tan^2 \theta}{3 \tan \theta - \tan^3 \theta} \]
\[ \Rightarrow \tan^2 \theta = \frac{3y - 1}{y - 3} > 0 \]
\[ \Rightarrow y = \left( -\infty, \frac{1}{3} \right) \cup (3, \infty) \]
\[ \therefore \] statement-1 and statement-2 both are true and statement-2 explains statement-1

22. \[ \cos x \cdot \sin y = 1 \]
\[ \Rightarrow \] Either \( \cos x = 1 \) and \( \sin y = 1 \) or \( \cos x = -1 \) and \( \sin y = -1 \)
\[ \Rightarrow (x, y) = \left( 0, \frac{\pi}{2} \right), \left( 0, \frac{5\pi}{2} \right), \left( 2\pi, \frac{\pi}{2} \right), \left( 2\pi, \frac{5\pi}{2} \right) \] or \( (x, y) = \left( \frac{\pi}{2}, \frac{3\pi}{2} \right), \left( \frac{3\pi}{2}, \frac{3\pi}{2} \right) \)
\[ \therefore \] Number of pairs = 6

24. \[ \log_2 [\cos^2 (\alpha + \beta) + \cos^2 (\alpha - \beta) - \cos 2\alpha \cos 2\beta] \]
\[ = \log_2 [\cos^2 (\alpha + \beta) + 1 - \sin^2 (\alpha - \beta) - \cos 2\alpha \cos 2\beta] \]
\[ = \log_2 [1 + \cos 2\alpha \cos 2\beta - \cos 2\alpha \cos 2\beta] \]
\[ = \log_2 1 \]
\[ = 0 \]
30. \[
\frac{\sin 30}{2 \cos 20 + 1} = \frac{1}{2} \Rightarrow 2(3 \sin \theta - 4 \sin^2 \theta) = 2(1 - 2 \sin^2 \theta) + 1
\]
\[
\Rightarrow 8 \sin^2 \theta - 6 \sin \theta - 4 \sin^2 \theta + 3 = 0 \quad \Rightarrow (2 \sin \theta - 1)(4 \sin^2 \theta - 3) = 0
\]
\[
\Rightarrow \sin \theta = \frac{1}{2} \pm \frac{\sqrt{3}}{2}
\]
For \( \sin \theta = \pm \frac{\sqrt{3}}{2} \), \( 2 \cos 2 \theta + 1 = 0 \) so given equation becomes undefined.
\[
\therefore \sin \theta = \frac{1}{2} \quad \Rightarrow \theta = n \pi + (-1)^n \frac{\pi}{6}, \; n \in \mathbb{I}.
\]
32. \[
\sin x \cdot \sqrt{8 \cos^2 x} = 1 \Rightarrow \sin x |\cos x| = \frac{1}{\sqrt{2}} \quad \Rightarrow 2 \sin x |\cos x| = \frac{1}{\sqrt{2}}.
\]
33. \[
\left( \cos^2 x + \frac{1}{\cos^2 x} \right) (1 + \tan^2 2y) (3 + \sin 3z) = 4
\]
\[
\Rightarrow \cos^2 x + \frac{1}{\cos^2 x} = 2, \; 1 + \tan^2 2y = 1, \; 3 + \sin 3z = 2
\]
\[
\Rightarrow \cos^2 x = 1, \; \tan^2 2y = 0, \; \sin 3z = -1 \quad \Rightarrow x = n \pi, \; n \in \mathbb{I}.
\]
36. \[
\cos 20^\circ + 2 \sin 55^\circ - \sqrt{2} \sin 65^\circ
\]
\[
= \cos 20^\circ + 1 - \cos 110^\circ - \sqrt{2} \sin 65^\circ
\]
\[
= 1 + 2 \sin 65^\circ \sin 45^\circ - \sqrt{2} \sin 65^\circ
\]
\[
= 1 + 2 \sin 65^\circ \frac{1}{\sqrt{2}} - \sqrt{2} \sin 65^\circ = 1
\]
41. \[
\frac{3 \sin \theta - \sin 3 \theta}{1 + \cos \theta} + \frac{3 \cos \theta + \cos 3 \theta}{1 - \sin \theta} = 4 \sqrt{2} \cos \left( \theta + \frac{\pi}{4} \right)
\]
\[
\Rightarrow \frac{4 \sin^3 \theta}{1 + \cos \theta} + \frac{4 \cos^3 \theta}{1 - \sin \theta} = 4(\cos \theta - \sin \theta) \quad \Rightarrow \frac{\sin^3 \theta}{1 + \cos \theta} + \sin \theta = \cos \theta - \frac{\cos^3 \theta}{1 - \sin \theta}
\]
\[
\Rightarrow \frac{\sin^3 \theta + \sin \theta + \sin \theta \cos \theta}{1 + \cos \theta} = \frac{\cos \theta - \cos \theta \sin \theta - \cos^3 \theta}{1 - \sin \theta}
\]
\[
\Rightarrow \frac{\sin \theta (\sin^2 \theta + 1 + \cos \theta)}{1 + \cos \theta} = \frac{\cos \theta \sin^2 \theta - \sin \theta \cos \theta}{1 - \sin \theta}
\]
\[
\Rightarrow \frac{\sin \theta (\sin^2 \theta + \cos \theta + 1)}{1 + \cos \theta} = \frac{- \sin \theta \cos \theta (- \sin \theta + 1)}{1 - \sin \theta} \quad \Rightarrow \text{either } \sin \theta = 0 \; \text{ or } \frac{\sin^2 \theta + \cos \theta + 1}{1 + \cos \theta} = - \cos \theta
\]
\[
\Rightarrow 0 = n \pi \; \text{ or } \sin^2 \theta + \cos \theta + 1 = - \cos \theta \; \text{ or } \cos \theta = -1
\]
\[
\Rightarrow 0 = n \pi \; \Rightarrow \theta = 2n \pi \; \text{ or } (2n + 1) \pi
\]
But at \( \theta = (2n + 1) \pi \), \( 1 + \cos \theta = 0 \)
\[
\therefore \theta = (2n + 1) \pi \quad \therefore \theta = 2n \pi.
\]
42. \[
\sqrt{6 - \cos x - 7 \sin^2 x} + \cos x = 0
\]
\[
\Rightarrow \sqrt{7 \cos^2 x - \cos x - 1} + \cos x = 0 \quad \Rightarrow \sqrt{7 \cos^2 x - \cos x - 1} = - \cos x
\]
\[
\text{(so } \cos x \leq 0) \Rightarrow 7 \cos^2 x - \cos x - 1 = \cos^2 x
\]
\[
\Rightarrow 6 \cos^2 x - \cos x - 1 = 0 \quad \Rightarrow \cos x = \frac{1}{2}, \; \frac{-1}{3}
\]
But \( \cos x \leq 0 \quad \therefore \quad \cos x = -\frac{1}{3} \)

\[ \Rightarrow \cos x = -\cos \alpha \quad \text{where} \quad \cos \alpha = \frac{1}{3} \Rightarrow \cos x = \cos(\pi - \alpha) \Rightarrow x = 2n\pi \pm (\pi - \alpha). \]

43. \( x^3 + x^2 + 4x + 2 \sin x = 0 \)
\[ \Rightarrow x^3 + x^2 + 4x = -2 \sin x \quad \text{...(1)} \]

when \( x = 0, \) then \( 0 = 0 \quad \therefore \quad x = 0 \) is the solution

when \( x \in [0, \pi), \) \( x^3 + x^2 + 4x > 0 \) whereas \(-2 \sin x < 0 \)

\( \therefore \) no solution for \( x \in (0, \pi) \)

when \( x \in [\pi, 2\pi], \) \( x^3 + x^2 + 4x \geq \pi^3 + \pi^2 + 4\pi > 2 \)

whereas \(-2 \sin x \leq 2 \)

\( \therefore \) no solution for \( [\pi, 2\pi] \)

so given equation has only one solution in \([0, 2\pi]\) and that solution is \( x = 0. \)

**EXERCISE # 4**

**PART - I**

1. Clearly \( \theta = 30^\circ \) and \( \phi \in (60^\circ, 90^\circ) \)

   Hence \( \theta + \phi \) lies in \((90^\circ, 120^\circ)\).

2. Let \( y = 2 \sin t \)

   \[ y = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1} \]

   \( (3y - 5) x^2 - 2x (y - 1) - (y + 1) = 0 \)

   \( x \in \mathbb{R} - \left\{1, -\frac{1}{3}\right\} \)

   \( \therefore \quad D \geq 0 \quad \Rightarrow \quad y^2 - y - 1 \geq 0 \)

   \( \therefore \quad y \geq 1 + \frac{\sqrt{5}}{2} \quad \text{or} \quad y \leq 1 - \frac{\sqrt{5}}{2} \)

   \( \Rightarrow \quad \sin t \geq 1 + \frac{\sqrt{5}}{4} \quad \text{or} \quad \sin t \leq 1 - \frac{\sqrt{5}}{4} \)

   \( \therefore \quad \text{range of } t \text{ is } \left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right] \)

3. \( \triangle O_1BD, \quad \frac{BD}{O_1D} = \cot 30^\circ \)

   \( \Rightarrow \quad BD = \sqrt{3} \quad \text{similarly } EC = \sqrt{3} \)

   \( \Rightarrow \quad BC = AB = AC = 2 + 2\sqrt{3} \)

   area of \( \triangle ABC = \frac{\sqrt{3}}{4} (2 + 2\sqrt{3})^2 = \frac{\sqrt{3}}{4} (1 + 3 + 2\sqrt{3}) 4 = 6 + 4\sqrt{3} \) sq. unit

4. \[ \alpha - \beta = 0, \quad -2\pi \quad \text{or} \quad 2\pi \]

   \( \alpha - \beta = 0 \quad \Rightarrow \quad \alpha = \beta \quad \Rightarrow \quad \cos 2\beta = \frac{1}{e} \)
This is true for '4' value of 'α', 'β'.

If $\alpha - \beta = -2\pi$ \Rightarrow $\alpha = -\pi$ and $\beta = \pi$ and $\cos(\alpha + \beta) = 1$ \Rightarrow (No solution)
similarly if $\alpha - \beta = 2\pi$ \Rightarrow $\alpha = \pi$ and $\beta = -\pi$ again no solution results.

5. \[ \theta \in \left(0, \frac{\pi}{4}\right) \]

\[ \therefore \tan \theta \uparrow \text{in } \theta \in \left(0, \frac{\pi}{4}\right) \text{ and } 0 < \tan \theta < 1 \]

\[ \cot \theta \downarrow \text{in } \theta \in \left(0, \frac{\pi}{4}\right) \text{ and } \cot \theta > 1 \]

Let $\tan \theta = 1 - \lambda_1$ and $\cot \theta = 1 + \lambda_2$ where $\lambda_1$ and $\lambda_2$ are very small and positive, then

\[ t_1 = (1 - \lambda_1)^{1 + \lambda_2}, t_2 = (1 - \lambda_1)^{1 - \lambda_2}, t_3 = (1 + \lambda_2)^{1 + \lambda_2}, t_4 = (1 + \lambda_2)^{1 - \lambda_2} \]

\[ \therefore \quad t_4 > t_3 > t_1 > t_2 \]

6. $2\sin^2 \theta - 5\sin \theta + 2 > 0$

\[ \Rightarrow (\sin \theta - 2)(2\sin \theta - 1) > 0 \]

\[ \Rightarrow \sin \theta < \frac{1}{2} \quad [\therefore -1 \leq \sin \theta \leq 1] \]

From graph, we get $\theta \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

7. $2\sin^2 \theta - \cos 2\theta = 0$ ............(i)

\[ \Rightarrow \sin \theta = \pm \frac{1}{2} \]

$2\cos^2 \theta - 3\sin \theta = 0$ ............(ii)

$-2\sin^2 \theta - 3\sin \theta + 2 = 0$

\[ \sin \theta = \frac{1}{2}, -2 \]

So $\sin \theta = \frac{1}{2}$ is the only solution

at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

8. \[ \frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5} \]

\[ \Rightarrow \frac{\sin^4 x}{2} + \frac{(1 - \sin^2 x)^2}{3} = \frac{1}{5} \]

\[ \Rightarrow 5 \sin^4 x - 4 \sin^2 x + 2 = \frac{6}{5} \]

\[ \Rightarrow (5 \sin^2 x - 2)^2 = 0 \]

\[ \Rightarrow \tan^2 x = \frac{2}{3} \]

and \[ \frac{\sin^6 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125} \]

9. \[ f(\theta) = \frac{1}{\sin^2 \theta + 3\sin \theta \cos \theta + 5\cos^2 \theta} = \frac{1 - \cos 2\theta}{2 + \frac{3}{2}\sin 2\theta + \frac{5(1 + \cos 2\theta)}{2}} = \frac{2}{6 + 3\sin 2\theta + 4\cos 2\theta} \]

\[ \therefore \quad f(\theta)_{\max} = \frac{2}{6 - 5} = 2 \]
10. \[
\frac{1}{\sin \frac{\pi}{n}} - \frac{1}{\sin \frac{3\pi}{n}} = \frac{1}{\sin \frac{2\pi}{n}}
\]
\[
2\cos \frac{2\pi}{n} \sin \frac{\pi}{n} - \sin \frac{\pi}{n} \sin \frac{3\pi}{n} = \frac{1}{\sin \frac{\pi}{n}}
\]
\[
\sin \frac{4\pi}{n} = \sin \frac{3\pi}{n}
\]
\[
\frac{4\pi}{n} = (-1)^k \frac{3\pi}{n} + k\pi, \ k \in \mathbb{I}
\]
If \( k = 2m \) \( \Rightarrow \frac{\pi}{n} = 2m\pi \)
\[
\frac{1}{n} = 2m, \text{ not possible}
\]
If \( k = 2m + 1 \) \( \Rightarrow \frac{7\pi}{n} = (2m + 1)\pi \)
\( \Rightarrow \ n = 7, \ m = 0 \)

11. \[
\tan \theta = \cot 5\theta
\]
\[
\Rightarrow \frac{\sin \theta}{\cos \theta} = \frac{\cos 5\theta}{\sin 5\theta} \quad \Rightarrow \cos 6\theta = 0
\]
\( \Rightarrow 6\theta = (2n + 1)\frac{\pi}{2} \quad \Rightarrow \theta = (2n + 1)\frac{\pi}{12}; \ n \in \mathbb{I}
\]
\( \Rightarrow \theta = -\frac{5\pi}{12}, -\frac{\pi}{4}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12} \quad (1)
\]
\( \sin 2\theta = \cos 4\theta
\)
\( \Rightarrow \sin 2\theta = 1 - 2\sin^2 2\theta \quad \Rightarrow 2\sin^2 2\theta + \sin 2\theta - 1 = 0
\)
\( \Rightarrow \sin 2\theta = -1, \frac{1}{2} \quad \Rightarrow 2\theta = (4m - 1)\frac{\pi}{2}, p\pi + (-1)^m \frac{\pi}{6}
\]
\( \Rightarrow \theta = (4m - 1)\frac{\pi}{4}, \frac{p\pi}{2} + (-1)^m \frac{\pi}{12}; \ m, \ p \in \mathbb{I}
\)
\( \Rightarrow \theta = -\frac{\pi}{4}, \frac{5\pi}{12} \quad (2)
\]
From (1) & (2)
\[
\theta \in \left\{ -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12} \right\}
\]
Number of solution is 3.

12. \[
P = \{\theta : \sin \theta - \cos \theta = \sqrt{2} \cos \theta\}
\]
\( \sin \theta = (\sqrt{2} + 1) \cos \theta \quad \Rightarrow \tan \theta = \sqrt{2} + 1
\]
\( \Rightarrow \theta = n\pi + \frac{3\pi}{8}; \ n \in \mathbb{I}
\]
\( Q = \{\theta : \sin \theta + \cos \theta = \sqrt{2} \sin \theta\} \quad : \ \cos \theta = (\sqrt{2} - 1) \sin \theta
\]
\( \Rightarrow \tan \theta = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 \quad \Rightarrow \theta = n\pi + \frac{3\pi}{8}; \ n \in \mathbb{I}
\]
\( \therefore \ P = Q \)
13. As \( \tan(2\pi - \theta) > 0, -1 < \sin\theta < -\frac{\sqrt{3}}{2}, \theta \in [0, 2\pi] \)
\[
\Rightarrow \frac{3\pi}{2} < \theta < \frac{5\pi}{3}
\]
Now \( 2\cos \theta (1 - \sin \phi) = \sin^2 \theta (\tan \theta/2 + \cot \theta/2) \cos \phi - 1 \)
\( \Rightarrow 2\cos \theta (1 - \sin \phi) = 2\sin \theta \cos \phi - 1 \)
\( \Rightarrow 2\cos \theta + 1 = 2\sin(\theta + \phi) \)
As \( \theta \in \left( \frac{3\pi}{2}, \frac{5\pi}{3} \right) \) \( \Rightarrow 2\cos \theta + 1 \in (1, 2) \)
\( \Rightarrow 1 < 2\sin(\theta + \phi) < 2 \)
\( \Rightarrow \frac{1}{2} < \sin(\theta + \phi) < 1 \)
As \( \theta + \phi \in [0, 4\pi] \)
\( \Rightarrow \theta + \phi \in \left[ \frac{3\pi}{2}, \frac{5\pi}{3} \right] \cup \left( \frac{2\pi}{3}, \frac{7\pi}{3} \right) \)
\( \therefore \) correct option is (A, C, D)

**PART - II**

1. \( u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \)
\( \Rightarrow u^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta + a^2 \sin^2 \theta + b^2 \cos^2 \theta + 2 \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \times \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \)
\( \Rightarrow u^2 = (a^2 + b^2) + 2 \sqrt{\left[ a^2 + (b^2 - a^2) \sin^2 \theta \right] \times \left[ a^2 + (b^2 - a^2) \cos^2 \theta \right]} \)
\( \Rightarrow u^2 = (a^2 + b^2) + 2 \sqrt{a^4 + 2a^2 b^2 - a^4 + b^4 - 2a^2 b^2} \sin^2 \theta \cos^2 \theta \)
\( \Rightarrow u^2 = (a^2 + b^2) + 2 \sqrt{a^2 b^2 + \left( \frac{b^2 - a^2}{2} \right)^2} \sin^2 \theta \).
\( \therefore \) min\((u^2) = a^2 + b^2 + 2ab = (a+b)^2 \)
and max\((u^2) = a^2 + b^2 + \left( a^2 + b^2 \right) = 2\left( a^2 + b^2 \right) \)
Now, max\((u^2) - \text{min}(u^2) = (a-b)^2 \)

2. \( \sin \beta = -\frac{21}{65} \) and \( \cos \beta = -\frac{27}{65} \)

squaring and adding, we get
\( \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta + \cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta \)
\[
= \left( -\frac{21}{65} \right)^2 + \left( -\frac{27}{65} \right)^2
\]
\( \Rightarrow 2 + 2 \cos (\alpha - \beta) = \frac{1170}{4225} \)
\( \Rightarrow \cos^2 \left( \frac{\alpha - \beta}{2} \right) = \frac{1170}{4 \times 4225} = \frac{9}{130} \)
\( \Rightarrow \cos \left( \frac{\alpha - \beta}{2} \right) = \frac{-3}{\sqrt{130}} \) \( \therefore \) \( \pi < \alpha - \beta < 3\pi \) \( \Rightarrow \frac{\pi}{2} < \left( \frac{\alpha - \beta}{2} \right) < \frac{3\pi}{2} \)
3. \[ \tan \frac{P}{2} \text{ and } \tan \frac{Q}{2} \] are the roots of equation \[ ax^2 + bx + c = 0 \]

\[ \therefore \tan \frac{P}{2} + \tan \frac{Q}{2} = -\frac{b}{a} \text{ and } \tan \frac{P}{2} \tan \frac{Q}{2} = \frac{c}{a} \]

\[ \frac{P}{2} + \frac{Q}{2} + \frac{R}{2} = \frac{\pi}{2} \quad (\because P + Q + R = \pi) \]

\[ \Rightarrow \frac{P + Q}{2} = \frac{\pi}{2} - \frac{R}{2} \]

\[ \Rightarrow \frac{P + Q}{2} = \frac{\pi}{4} \quad (\because \angle R = \frac{\pi}{2}) \]

\[ \Rightarrow \tan \left( \frac{P + Q}{2} \right) = 1 \Rightarrow \frac{\tan \frac{P}{2} + \tan \frac{Q}{2}}{1 - \tan \frac{P}{2} \tan \frac{Q}{2}} = 1 \Rightarrow -\frac{b}{a - c} = 1 \Rightarrow c = a + b \]

4. \[ \cos x + \sin x = \frac{1}{2} \]

\[ \therefore \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2} + \frac{2 \tan x/2}{1 + \tan^2 x/2} = \frac{1}{2} \]

Let \( \tan \frac{x}{2} = t \)

\[ \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2} = \frac{1}{2} \Rightarrow 3t^2 - 4t - 1 = 0 \]

\[ \therefore t = \frac{2 \pm \sqrt{7}}{3} \]

as \( 0 < x < \pi \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{2} \) \[ \therefore \tan \frac{x}{2} \text{ is positive} \]

\[ \therefore t = \tan \frac{x}{2} = \frac{2 + \sqrt{7}}{3} \]

Now \( \tan x = \frac{2 \tan x/2}{1 - \tan^2 x/2} = \frac{2t}{1 - t^2} \]

\[ \Rightarrow \tan x = \frac{2 \left( \frac{2 + \sqrt{7}}{3} \right)}{1 - \left( \frac{2 + \sqrt{7}}{3} \right)^2} = -\left( \frac{4 + \sqrt{7}}{3} \right) \]

5. Given equation is \( 2 \sin^2 x + 5 \sin x - 3 = 0 \)

\[ \Rightarrow (2 \sin x - 1) (\sin x + 3) = 0 \]

\[ \Rightarrow \sin x = \frac{1}{2} \quad (\because \sin x \neq -3) \]

\[ y \]

\[ \text{It is clear from figure that the curve intersect the line at four points in the given interval. Hence, number of solutions are 4.} \]

6. Given, \[ \cos x + \sin x = \frac{1}{2} \]

\[ \therefore \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2} + \frac{2 \tan x/2}{1 + \tan^2 x/2} = \frac{1}{2} \]
Let \( \tan \frac{x}{2} = t \quad \Rightarrow \quad \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = \frac{1}{2} \)

\[ \Rightarrow \quad 3t^2 - 4t - 1 = 0 \quad \Rightarrow \quad t = \frac{2 \pm \sqrt{7}}{3} \]

As \( 0 < x < \pi \) \( \Rightarrow \quad 0 < \frac{x}{2} < \frac{\pi}{2} \) \quad \therefore \quad \tan \frac{x}{2} \) is positive.

\[ \therefore \quad t = \tan \frac{x}{2} = \frac{2 + \sqrt{7}}{3} \]

Now, \( \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = \frac{2t}{1-t^2} \quad \Rightarrow \quad \tan x = \frac{2 \left( \frac{2 + \sqrt{7}}{3} \right)}{1 - \left( \frac{2 + \sqrt{7}}{3} \right)^2} \]

\[ \Rightarrow \quad \tan x = -\frac{3 \left( \frac{2 + \sqrt{7}}{3} \right)}{1 + 2\sqrt{7}} \times \frac{1 - 2\sqrt{7}}{1 - 2\sqrt{7}} \quad \Rightarrow \quad \tan x = -\left( \frac{4 + \sqrt{7}}{3} \right) \).

7. \( 2(\cos (\beta - \gamma) + \cos (\gamma - \alpha) + \cos (\alpha - \beta)) + 3 = 0 \)
\( (\cos \alpha + \cos \beta + \cos \gamma)^2 + (\sin \alpha + \sin \beta + \sin \gamma)^2 = 0 \)
\( \sum \cos \alpha = 0 = \sum \sin \alpha \)

8. \( \tan 2\alpha = \tan ((\alpha + \beta) + (\alpha - \beta)) = \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)} = \frac{3}{4} \cdot \frac{5}{12} = \frac{9 + 5}{48 - 15} = \frac{56}{33} \)

Hence correct option is (1)

9. \( A = \sin^2 x + \cos^4 x \)
\( = \sin^2 x + (1 - \sin^2 x)^2 \)
\( = \sin^4 x - \sin^2 x + 1 \)
\( = \left( \sin^2 x - \frac{1}{2} \right)^2 + \frac{3}{4} \)
\( \leq A \leq 1 \)

10. \( 3\sin P + 4 \cos Q = 6 \) \quad \ldots (i)
\( 4 \sin Q + 3 \cos P = 1 \) \quad \ldots (ii)

Squaring and adding (i) \& (ii) we get \( \sin (P + Q) = \frac{1}{2} \)

\[ \Rightarrow \quad P + Q = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \quad \Rightarrow \quad R = \frac{5\pi}{6} \text{ or } \frac{\pi}{6} \]

If \( R = \frac{5\pi}{6} \) then \( 0 < P, Q < \frac{\pi}{6} \)

\[ \Rightarrow \quad \cos Q < 1 \text{ and } \sin P < \frac{1}{2} \quad \Rightarrow \quad 3 \sin P + 4 \cos Q < \frac{11}{2} \)

So \( R = \frac{\pi}{6} \)
Section (A):

A-5. \[ \log_p \left( \frac{1}{p} \right)^n = \log_p \left( \frac{1}{p^n} \right) = -n \]

independent of \( p \).

A-7*. \[ N = 3 \log_{135} 3 - \log_{405} 3 \]

\[ = \left( \log_3 27 + \log_3 5 \right) \left( \log_{15} 5 \right) - \log_5 \log_3 405 \]

\[ = (3 + \log_3 5) (1 + \log_3 5) - \log_5 (81 \times 5) \]

\[ = (3 + \log_3 5) (1 + \log_3 5) - \log_5 (4 + \log_3 5) \]

\[ = 3. \]

A-8*. \[ \log_2 3 > 1, \log_{12} 10 < 1 \]

\[ \log_6 5 < 1, \log_7 8 > 1 \]

\[ \log_{16} 15 < 3, \log_2 9 > 3 \]

\[ \log_{10} 11 > 1 \]


Section (B):

B-4*. \[ \left( \log_5 x \right)^2 + \log_5 \frac{5}{x} = 1 \]

\[ \Rightarrow \left( \log_5 x \right)^2 + \log_5 5 - \log_5 x = 1 \]

\[ \Rightarrow \left( \log_5 x \right)^2 + \frac{1}{1 + \log_5 x} - \frac{\log_5 x}{1 + \log_5 x} = 1 \]

Let \( \log_5 x = t \)

\[ \Rightarrow t^2 + \frac{1}{1+t} - \frac{t}{1+t} = 1 \]

\[ \Rightarrow t^2 + t^2 + t - t = 1 + t \]

\[ t^3 + t^2 - 2t = 0 \]

\[ t(t^2 + t - 2) = 0 \]

\[ t(1 + 1) = 0 \]

\[ t = 0, 1, -2 \]

\[ \therefore x = 1, 5, \frac{1}{25} \]

B-6*. \[ \left[ \left( \log_3 x \right)^2 - \frac{9}{2} \log_3 x + 5 \right] = 3 \sqrt{3} \]

\[ \Rightarrow \left( \log_3 x \right)^2 - \frac{9}{2} \log_3 x + 5 = \frac{3}{2} \log_3 3 \sqrt{3} \]

Let \( \log_3 x = t \)

\[ \Rightarrow t^2 - \frac{9}{2} t + 5 = \frac{3}{2} \]

\[ \Rightarrow 2t^2 - 9t + 10t - 3 = 0 \]

\[ t = 1 \] satisfies it

\[ 2t^2 - 9t + 10t - 3 = 2t^2(t - 1) - 7t(t - 1) + 3(t - 1) \]

\[ = (t - 1) \left( 2t^2 - 7t + 3 \right) \]

\[ = (t - 1) \left( 2t - 1 \right) (t - 3) \]
⇒ $t = 1$  $t = \frac{1}{2}$  $t = 3$
⇒ $\log_3 x = 1$  $\log_3 x = \frac{1}{2}$  $\log_3 x = 3$
⇒ $x = 3$  $x = 3^{1/2}$  $x = 27$

B-9. Number of digits in integral part = number of digit in $60^{12}$ before decimal
$P = 60^{12}$
$\log P = \log 60^{12} = 12 \log 60 = 12[\log 6 + 1] = 12[\log 2 + \log 3 + 1]$
number of digits in integral part = 22

B-10. $\log_{16} x = \frac{3}{4}$
⇒ $x = 16^{3/4}$  ⇒  $x = 8$

Section (C) :

C-3. $\log_{1-\varepsilon} (x - 2) \geq -1$
$\Rightarrow x > 2$  ....................(1)
(i) When $0 < 1 - x < 1$  ⇒  $0 < x < 1$
So no common range comes out.
(ii) When $1 - x > 1$  ⇒  $x < 0$ but $x > 2$
here, also no common range comes out., hence no solution.
Finally, no solution

C-6. $\sqrt[3]{(x - 8)(2 - x)} \geq 0$
$\log_{0.3} \left[ \frac{10}{7} \log_2 (5/2) \right] \geq 0$

For $\sqrt[3]{(x - 8)(2 - x)}$ to be defined
(i) $(x - 8)(2 - x) \geq 0$
$(x - 2)(x - 8) \leq 0$  ⇒  $2 \leq x \leq 8$
Now  Let say $y = \log_{0.3} \left[ \frac{10}{7} \log_2 (5/2) \right] = \log_{0.3} \left[ \frac{10}{7} \log_2 (5/2) \right]$
Let  $y < 0$  (assume)
then $\log_{0.3} \left[ \frac{10}{7} \log_2 (5/2) \right] < 0$
⇒ $\frac{10}{7} \log_2 (5/2) > 1$  ⇒  $\log_2 (5/2) > \frac{7}{10}$  ⇒  $\frac{5}{2} > 2^{(7/10)}$ which is true
So  $y < 0$
special denominator is – ve and numerator is +ve, so inequality is not satisfied,
thus $\sqrt[3]{(x - 8)(2 - x)} = 0$
$\Rightarrow x = 2, 8$  .....(i)
Now  $2^{1/3} > 31$
⇒ $(x - 3) > \log_2 31$  ⇒  $x > 3 + \log_2 2^{4.9}$ (approx) ⇒  $x > 7.9$
⇒ $x = 8$

C-8. Domain $x^2 + 4x - 5 \geq 0$
⇒ $x \in (-\infty, -5] \cup [1, \infty)$
Case I : $x \in (-\infty, -5] \cup [1, 3)$
- ve < + ve  always true
∴  $x \in (-\infty, -5] \cup [1, 3)$  .... (1)
Case II :
$x \in [3, \infty)$  .. (i)
\[ x - 3 < \sqrt{x^2 + 4x - 5} \]

\[ \Rightarrow \quad x^2 - 6x + 9 < x^2 + 4x - 5 \]

\[ \Rightarrow \quad x > \frac{7}{5} \quad \text{... (ii)} \]

(i) \( \cap \) (ii) \( x \in [3, \infty) \quad \text{... (2)} \)

(1) \( \cup \) (2) \( x \in (-\infty, -5] \cup [1, \infty) \) Ans. (A)

**Section (D) :**

D-3. \( z = \frac{\pi}{4} (1+i)^4 \) \left[ \frac{1+\pi+\pi+1}{(\sqrt{\pi}+i)(1+\sqrt{\pi}i)} \right] = \frac{\pi}{4} (1+i)^4 \left( \frac{2}{i} \right) \)

\[ = \frac{\pi}{2} (1+i)^4 = \frac{\pi}{2} 4e^{ix} = 2\pi e^{ix/2} \]

|z| = 2\pi \quad \text{amp} (z) = \frac{\pi}{2}

\[ \left( \frac{|z|}{\text{amp}(z)} \right) = \frac{2\pi}{\frac{\pi}{2}} = 4 \quad \text{(D)} \]

D-6*. \[ |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \]

\[ z_1z_2 + \bar{z}_1\bar{z}_2 = 0 \]

\[ \Rightarrow \quad \frac{z_1}{z_2} = -\frac{\bar{z}_1}{\bar{z}_2} \]

\[ \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} = 0 \]

\[ \Rightarrow \quad \frac{z_1}{z_2} \text{ is purely imaginary} \]

so \( \text{amp} \left( \frac{z_1}{z_2} \right) \) is may be \( \frac{\pi}{2} \) or \( -\frac{\pi}{2} \)

D-8.

D-9. \( z^{1/3} = a - ib \)

\[ z = (a - ib)^3 \]

\[ x + iy = (a^3 - 3ab^2) + i(b^3 - 3a^2b) \]

\[ \Rightarrow \quad \frac{x}{a} = a^2 - 3b^2 \quad \frac{y}{b} = b^2 - 3a^2 \]

\[ \frac{x}{a} - \frac{y}{b} = 4(a^2 - b^2) \quad k = 4 \]

**EXERCISE # 2**

1. \( (i) \quad \left( 5^{\log_{10} 7} + \frac{1}{\log_{10} \left( \frac{1}{0.1} \right)} \right)^{1/3} = (7 + 1)^{1/3} = 2 \)

\( (ii) \quad \log_{3/4} \log_{5} \left( 8^{1/2} \right)^{1/2} = \log_{3/4} \log_{2} (2)^{3/4} = 1 \)

**Solutions (XI) # 30**
(iii) \( \left( \frac{1}{49} \right)^{1 + \log_7^2} = (7^{-2})^{\log_7 2} = (7^{-2})^{\log_{14} 7} = \log_{196} 1 \) & \( 5^{\log_{7/5} 7} = 5^{\log_7 7} = 7 \)

\[ \therefore \quad 7 + \frac{1}{196} \]

(iv) \( 7^{\log_7 5} + 3^{\log_7 7} \) & \( 5^{\log_7 7} - 7^{\log_3 3} \)

\[ = 7^{\log_7 5} + 3^{\log_7 7} - 7^{\log_3 3} \]

\{using property \( a^{\log_a b} = b^{\log_b a} \}\]

\[ = 0 \]

5. (i) \( \log_{10} (x^2 - 12x + 36) = 2 \)

\( (i) \quad x^2 - 12x + 36 > 0 \) \\( \Rightarrow \quad x \in \mathbb{R} - \{6\} \)

\( (ii) \quad x^2 - 12x + 36 = 100 \) \\( \Rightarrow \quad (x - 16)(x + 4) = 0 \)

\[ \therefore \quad x = 16, -4. \]

(ii) \( \log_4 \log_3 \log_2 x = 0 \)

\[ \Rightarrow \quad \log_3 \log_2 x = 1 \]

\[ \Rightarrow \quad \log_2 x = 3 \]

\[ \Rightarrow \quad x = 2^3 \]

\[ \Rightarrow \quad x = 8. \]

(iii) \( \log_3 \left( \frac{1}{2} + 9^x \right) = 2x \)

\[ \Rightarrow \quad \log_9 x + \frac{1}{2} + 9^x = 9^x \]

\[ \Rightarrow \quad \log_9 x = -\frac{1}{2} \]

\[ \Rightarrow \quad x = 9^{-1/2} \]

\[ \Rightarrow \quad x = \frac{1}{3} \]

(iv) \( 2 \log_4 (4 - x) = 4 - \log_2 (-2 - x) \)

\( (i) \quad 4 - x > 0 \) \\( \Rightarrow \quad x < 4 \)

\( (ii) \quad -2 - x > 0 \) \\( \Rightarrow \quad x < -2 \)

\( (iii) \quad \log_2 (4 - x) = 4 - \log_2 (-2 - x) \)

\[ \Rightarrow \quad \log_2 (4 - x) (-2 - x) = 4 \]

\[ \Rightarrow \quad (4 - x)(-2 - x) = 16 \]

\[ \Rightarrow \quad x^2 - 2x - 24 = 0 \]

\[ \Rightarrow \quad (x - 6)(x + 4) = 0 \]

\[ \therefore \quad x = 6 \text{ (not possible)} , \quad x = -4. \]

(v) \( \log_{10} x + \log_{10} x^2 = \log_{10} 2 - 1 \)

\[ \log_{10} x + 2 \log_{10} x + 1 = \log_{10} 2 \]

\[ \log_{10} x + 1 = \log_{10} 2 \]

\[ \Rightarrow \quad x = \frac{1}{20} \quad \text{and} \quad \frac{1}{5} \]

(vi) \( \log_4 \log_2 x + \log_2 \log_4 x = 2 \)

\[ \Rightarrow \quad \frac{1}{2} \log_2 (2 \log_4 x) + \log_2 \log_4 x = 2 \]

\[ \Rightarrow \quad \frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 \log_4 x + \log_2 \log_4 x = 2 \]

\[ \Rightarrow \quad \frac{3}{2} \log_2 \log_4 x = \frac{3}{2} \]

\[ \Rightarrow \quad \log_2 \log_4 x = 1 \]

\[ \Rightarrow \quad \log_4 x = 2 \]

\[ \Rightarrow \quad x = 4^2 \quad \Rightarrow \quad x = 16. \]

(vii) \( \frac{\log x + 5}{3} = 10^{5 + \log x} \)

\[ \left( \frac{\log x + 5}{3} \right) \log x = 5 + \log x \]
\[
\log^2 x + 2 \log x - 15 = 0 \\
(\log x + 5)(\log x - 3) = 0 \\
\log x = -5, \log x = 3 \\
x = 10^{-5}, \ x = 10^3.
\]

(viii) Domain \( x - 1 > 0 \) and \( x + 1 > 0 \) and \( y - x > 0 \)
\[
x > 1 \quad x > -1 \quad x < 7 \\
\Rightarrow \ x \in (1, 7) \quad \ldots \quad (i)
\]
\[- \log_2 (x - 1) - \log_2 (x + 1) = 1 + \log_2 (7 - x) \]
\[- \log_2 (x^2 - 1) + \log_2 (7 - x)^2 = 1 \]
\[
\log_2 \left( \frac{(7-x)^2}{x^2-1} \right) = 1 \quad \Rightarrow \quad \frac{(7-x)^2}{x^2-1} = 2 \\
\Rightarrow \ x^2 + 14x - 51 = 0 \\
(x + 17) (x - 3) = 0 \\
x = 3 \text{ or } x = -17 \ (\text{rejected}) \\
x = 3
\]

6. (a) \( \log_{10} 2 = 0.3010 \cdot \log_{10} 3 = 0.4771 \)
let \( x = 6^{15} \)
\[
\log_{10} x = 15 \log_{10} 6 \\
= 15(\log_{10} 2 + \log_{10} 3) \\
= 15(0.3010 + 0.4771) \\
= 11.6715 \\
\Rightarrow \ \text{characteristic of } 6^{15} \text{ is } 11 \\
\Rightarrow \ \text{number of digits in } 6^{15} \text{ is } 12.
\]
(b) let \( x = 3^{-100} \)
\[
\log_{10} x = -100 \log_{10} 3 \\
= -47.71 \\
\Rightarrow \ \text{number of zeroes immediately after the decimal in } 3^{-100} \text{ is } 47.
\]

10. (i) \[
\log_5 \frac{4x+6}{x} \geq 0 \\
\frac{4x+6}{x} > 0 \\
\Rightarrow \ x \in \left(-\infty, -\frac{3}{2}\right) \cup (0, \infty) \quad \ldots \quad (i)
\]
(ii) \[
\log_2 (4^x - 2.2^x + 17) > 5 \\
4^x - 2.2^x + 17 > 0 \\
(2^x)^2 - 2.2^x + 17 > 0 \\
\Rightarrow \ (2^x)^2 - 2.2^x - 15 > 0 \\
\Rightarrow \ 2^x < -3 \quad \text{or} \quad 2^x > 5 \\
\Rightarrow \ x \in (-\infty, \phi) \quad \text{or} \quad x > \log_2 5 \\
\Rightarrow \ x \in (\log_2 5, \infty)
\]
(iii) \[
(\log x)^2 - \log x - 2 \geq 0 \\
x > 0 \\
(\log x - 2)(\log x + 1) \geq 0 \\
\Rightarrow \ \log x \leq -1 \quad \text{or} \quad \log x \geq 2 \\
\Rightarrow \ x \leq \frac{1}{e} \quad \text{or} \quad x \geq 100 \quad \ldots \quad (ii)
\]
\[
(i) \cap (ii) \Rightarrow \ x \in \left[0, \frac{1}{e}\right] \cup [100, \infty)
\]
(iv) \[\log_{0.5}((x + 5)^2) > \log_{1/2}(3x - 1)^2\]
\[(x + 5)^2 > 0 \quad \Rightarrow \quad x \in \mathbb{R} - \{-5\} \quad \ldots \ldots \text{(i)}\]
\[(3x - 1)^2 > 0 \quad \Rightarrow \quad x \in \mathbb{R} - \left\{\frac{1}{3}\right\} \quad \ldots \ldots \text{(ii)}\]
\[(x + 5)^2 < (3x - 1)^2\]
\[\Rightarrow \quad 8x^2 - 16x - 24 > 0 \quad \Rightarrow \quad x^2 - 2x - 3 > 0\]
\[\Rightarrow \quad (x - 3)(x + 1) > 0 \quad \Rightarrow \quad x \in (-\infty, -1) \cup (3, \infty) \quad \ldots \ldots \text{(iii)}\]
(i) \(\cap\) (ii) \(\cap\) (iii) gives
\[(-\infty, -5) \cup (-5, -1) \cup (3, \infty)\]

(v) \[\log_{3x^2+1}2 \leq \frac{1}{2}\]
\[3x^2 + 1 > 1 \quad \Rightarrow \quad x^2 > 0 \quad \Rightarrow \quad x \in \mathbb{R} - \{0\}\]
\[2 < (3x^2 + 1)^{1/2}\]
\[\Rightarrow \quad 3x^2 + 1 > 4 \quad \Rightarrow \quad (x - 1)(x + 1) > 0\]
\[\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)\]

(vi) \[\log_{x^2}(x + 2) < 1\]
\[x + 2 > 0 \quad \Rightarrow \quad x > -2\]

Case-I : when \[0 < x^2 < 1\]
then \[x + 2 > x^2 \quad \Rightarrow \quad x^2 - x - 2 < 0\]
\[x \in (-1, 0) \cup (0, 1)\]

Case-II : \[x^2 > 1\]
\[x + 2 < x^2 \quad \Rightarrow \quad x^2 - x - 2 > 0\]
\[x \in (-\infty, -1) \cup (2, \infty)\]

Hence, \[x \in (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (2, \infty)\]

11. (i) \[\frac{\sqrt{2x - 1}}{x - 2} < 1\]
Case-I: \[-2 < 0 \quad \Rightarrow \quad x < 2 \quad \ldots \ldots \text{(i)}\]
\[2x - 1 > (x - 2)^2\]
\[x \in (-\infty, 1) \cup (5, \infty) \quad \ldots \ldots \text{(ii)}\]
\[x \in \text{(i)} \cap \text{(ii)} \quad x \in (-\infty, 1) \quad \ldots \ldots \text{(A)}\]
Case-II: \[-2 > 0 \quad \Rightarrow \quad x > 2 \quad \ldots \ldots \text{(iii)}\]
\[2x - 1 < (x - 2)^2\]
\[2x - 1 < x^2 - 4x + 4\]
\[x^2 - 6x + 5 > 0\]
\[\Rightarrow \quad x \in (-\infty, 1) \cup (5, \infty) \quad \ldots \ldots \text{(iv)}\]
\[x \in \text{(iii)} \cap \text{(iv)} \quad x \in (5, \infty) \quad \ldots \ldots \text{(B)}\]
\[x \in \text{(A)} \cup \text{(B)} \quad x \in (-\infty, 1) \cup (5, \infty)\]

(ii) \[x < \sqrt{1 - |x|}\]
Case-I: \[x < 0 \quad \ldots \ldots \text{(i)}\]
\[1 - |x| \geq 0 \quad \Rightarrow \quad 1 + x \geq 0 \quad \Rightarrow \quad x \geq -1 \quad \ldots \ldots \text{(ii)}\]
\[x \in \text{(i)} \cap \text{(ii)} \quad x \in [-1, 0) \quad \ldots \ldots \text{(A)}\]
Case-II: \[x \geq 0 \quad \ldots \ldots \text{(i)}\]
\[1 - x \geq 0 \quad \Rightarrow \quad x \leq 1 \quad \ldots \ldots \text{(ii)}\]
\[x^2 < 1 - x\]
\[\Rightarrow \quad x^2 + x < 1 \quad \Rightarrow \quad x^2 + x + \frac{1}{4} < \frac{5}{4}\]
\[\Rightarrow \quad \left(\frac{x + 1}{2}\right)^2 < \frac{5}{4}\]
\[-\frac{1 - \sqrt{5}}{2} < x < \frac{\sqrt{5} - 1}{2} \quad \ldots \ldots \text{(iii)}\]
\[ x \in (i) \cap (ii) \cap (iii) \quad \Rightarrow \quad x \in \left[ 0, \frac{\sqrt{5} - 1}{2} \right] \quad \text{(B)} \]

\[ x \in (A) \cup (B) \quad \Rightarrow \quad x \in \left[ -1, \frac{\sqrt{5} - 1}{2} \right) \]

(iii) \[ \sqrt{x^2 - 6x + 8} \leq \sqrt{x + 1} \]
Domain \( x + 1 \geq 0 \implies x \geq -1 \)
\[ x^2 - 6x + 8 \geq 0 \implies (x - 2)(x - 4) \geq 0 \]
\[ \Rightarrow x \leq 2 \text{ or } x > 4 \]
\[ \Rightarrow \text{ Domain } \Rightarrow x \in [-1, 2] \cup [4, \infty) \]
squaring \( x^2 - 6x + 8 \leq x + 1 \implies x^2 - 7x + 7 \leq 0 \)
\[ \left( x - \frac{7}{2} \right)^2 - \frac{21}{4} \leq 0 \implies x \in \left[ \frac{7 - \sqrt{21}}{2}, \frac{7 + \sqrt{21}}{2} \right] \]

(iv) \[ \sqrt{8 + 2x - x^2} > 6 - 3x \]
(a) 8 + 2x - x^2 \geq 0 \quad \Rightarrow \quad x \in [-2, 4] \quad \text{... (i)}

Case - I

when (i) \( 6 - 3x \geq 0 \) \quad \Rightarrow \quad x \leq 2 \quad \text{... (ii)}

so \( 8 + 2x - x^2 > 36 + 9x^2 - 36x \)
\[ \Rightarrow 10x^2 - 38x + 28 < 0 \]
\[ \Rightarrow 5x^2 - 19x + 14 < 0 \]
\[ \Rightarrow (5x - 14)(x - 1) < 0 \]
\[ x \in \left( 1, \frac{14}{5} \right] \quad \text{... (iii)} \]

by (1) and (2) and (3)
\[ x \in (1, 2] \]
Case - II

\[ 6 - 3x < 0 \quad \Rightarrow \quad x > 2 \]
+ ve > -ve

so \( x > 2 \) \quad \text{... (iv)}
by (1) and (4)
\[ x \in (2, 4] \]
so by case (1) and (2) \( x \in (1, 4] \)

(v) \[ x^2 - 7x + 10 \geq 0 \text{ and } 14x - 20 - 2x^2 \geq 0 \]
\( (x - 2)(x - 5) \geq 0 \text{ and } (x - 2)(x - 5) \leq 0 \) \text{ ... (i)}

so \( x = 2 \) or \( x = 5 \)
now check for \( x = 2 \)

\[ 9 \log_4 \left( \frac{1}{4} \right) \geq -9 \]
\[ -9 \geq -9 \]
which is true hence \( x = 2 \) is a solution
now check \( x = 5 \)

\[ \frac{9}{2} \log \left( \frac{5}{8} \right) \geq -3 \]
\[ \log_2 \left( \frac{5}{8} \right) \geq -\frac{2}{3} \]
\( (1.6)^3 \leq 4 \)
4.096 \leq 4
which is false
so only solution is \( x = 2 \)
Domain $x > 0$
\[
\log_2^2 x + 2 \log_2 x \geq 0
\]
\[
\log_2 x (\log_2 x + 2) \geq 0
\]
\[
\begin{array}{ccc}
+ve & -ve & +ve \\
-2 & 0 & \infty
\end{array}
\]
\[
\log_2 x \leq -2 \text{ or } \log_2 x \geq 0
\]
\[
0 < x \leq \frac{1}{4} \text{ or } x \geq 1
\]
\[
x \in \left(0, \frac{1}{4}\right] \cup [1, \infty) \quad \ldots \ldots (i)
\]

Case-I

\[
4 - \log_2 x < 0
\]

positive < negative (false)

Case-II

\[
4 - \log_2 x \geq 0
\]

\[
\Rightarrow \log_2^2 x + 2 \log_2 x < (4 - \log_2 x)^2
\]

Let
\[
\log_2 x = t
\]

\[
t^2 + 2t < 2 (4 - t)^2
\]

\[
t^2 - 18t + 32 > 0
\]

\[
(t - 16)(t - 2) > 0
\]

\[
\log_2 x < 2 \cup \log_2 x > 16 \quad \text{(Rejected)}
\]

\[
x < 4
\]

...... (ii)

by (i) and (ii)

\[
x \in \left(0, \frac{1}{4}\right] \cup [1, 4)
\]

13. Square root of $7 + 2i = \pm \left[\sqrt{\frac{7^2 + 2^2}{2}} + i\sqrt{\frac{7^2 - 2^2}{2}}\right] = \pm(4 + 3i)

where $|7 + 2i| = 25$

15. (i) $z = \alpha$

$\alpha \in \mathbb{R}$

$\alpha^2 - (3 + i)\alpha + m + 2i = 0$

$\alpha^2 - 3\alpha + m = 0 \quad \& \quad -\alpha + 2 = 0$

$\alpha = 2$

$8 - 6 + m = 0 \quad \Rightarrow \quad m = -2$

(ii) If one root is $i$ then other is $-i$

Let forth root is $\alpha$.

\[
2\alpha = \frac{3}{2} \quad \Rightarrow \quad \alpha = \frac{3}{4}
\]

\[
-\frac{a}{2} = 2 + i + (-i) + \frac{3}{4} = \frac{11}{4}
\]

\[
a = -\frac{11}{2}
\]

20. (i) $z = 1 + e^{\frac{18\pi}{25}} = e^{\frac{9\pi}{25}} \left[ e^{\frac{9\pi}{25}} + e^{-\frac{9\pi}{25}} \right]$

\[
z = 2\cos\left(\frac{9\pi}{25}\right) e^{\frac{9\pi}{25}}
\]

\[
|z| = 2\cos\left(\frac{9\pi}{25}\right) \quad \text{Arg} \ z = \frac{9\pi}{25}
\]

(ii) $z = 2e^{i\frac{\pi}{6}} = 2e^{-i\frac{5\pi}{6}}$

\[
|z| = 2 \quad \text{Arg} \ z = -\frac{5\pi}{6}
\]
(iii) \[ |z| = \left( \sqrt{1 + \tan^2 1} \right)^2 = \sec^2 1 \]

Arg \( z \) = 2 Arg(tan 1 – i)

= 2 \left( \frac{1 - \pi}{2} \right) = 2 - \pi

(iv) \[ z = \frac{(i - 1)}{2 \sin \left( \frac{\pi}{5} \right) \sin \left( \frac{\pi}{5} \right) i + \cos \left( \frac{\pi}{5} \right)} \]

\[ |z| = \frac{\sqrt{2}}{2 \sin \left( \frac{\pi}{5} \right)} = \frac{1}{\sqrt{2}} \sec \left( \frac{\pi}{5} \right) \]

Arg(z) = \pi - \frac{\pi}{4} - \frac{\pi}{5} = \frac{11\pi}{20}

**EXERCISE # 3**

**PART - I**

1. \[ \sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1} \]

   squaring both sides

   \[(x + 1) + (x - 1) - 2 \sqrt{x^2 - 1} = 4x - 1 \]

   \[(1 - 2x) = 2 \sqrt{x^2 - 1} \]

   squaring both sides

   \[1 + 4x^2 - 4x = 4x^2 - 4 \]

   \[4x = 5 \Rightarrow x = \frac{5}{4} \]

   does not satisfy equation (i)

   \[\therefore \text{No solution} \]

2. \[ 2 \log_2 \log_2 x + \log_{1/2} \log_2 \left(2\sqrt{2}x\right) = 1 \]

   \[\Rightarrow \log_2 (\log_2 x)^2 - \log_2 \log_2 \left(2\sqrt{2}x\right) = 1 \]

   \[\Rightarrow \log_2 \frac{(\log_2 x)^2}{\log_2 \left(2\sqrt{2}x\right)} = 1 \]

   \[\Rightarrow \frac{3}{2} + \log_2 x = 2 \]

   Let \( \log_2 x = y \)

   \[\therefore y^2 - 2y - 3 = 0 \]

   \[\therefore y = 3, -1 \]

   but \( \log_2 x > 0 \)

   \[\therefore \log_2 x = -1 \text{ is not possible} \]

   \[\Rightarrow x = 8 \]

3. (a) \[ |z_1| = |z_2| = |z_3| = 1 \]

   \[z_2z_1 = z_2z_2 = z_3z_3 = 1 \]

   **Given**

   \[ 1 = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = \left| z_1 + z_2 + z_3 \right| = \left| z_1 + z_2 + z_3 \right| = 1 \]

   \[1 = |z_1 + z_2 + z_3| \]

   (b) \( -\theta = \arg (z) < 0 \)

   \[\arg (-z) = \pi - \theta \]

   \[\Rightarrow \arg (-z) - \arg (z) = \pi - \theta - (-\theta) = \pi \]

   Hence (A)
4. \[ \log_{3/4} \log_8 (x^2 + 7) + \log_{1/2} \log_{1/4} (x^2 + 7)^{-1} = -2 \]

\[ \Rightarrow \log_{3/4} \frac{1}{3} \log_2 (x^2 + 7) - \log_2 \frac{\log_2(x^2 + 7)}{2} = -2 \]

Let \( \log_2 (x^2 + 7) = t \)

\[ \Rightarrow \log_{3/4} \frac{1}{3} - \log_2 \frac{t}{2} + 2 = 0 \]

\[ \Rightarrow \log_{3/4} \frac{1}{4} = \log_2 \frac{t}{4} \]

\[ \Rightarrow \frac{1}{4} = 1 \Rightarrow t = 4 \]

\[ \therefore \log_2 (x^2 + 7) = 4 \]

This gives \( x = \pm 3 \).

5. \[ \frac{1}{2} \log_2 (x - 1) = \log_2 (x - 3) \]

\[ \sqrt{x - 1} = x - 3 \]

\( (x - 1) = x^2 - 6x + 9 \)

\( x^2 - 7x + 10 = 0 \)

\( (x - 5)(x - 2) = 0 \) but \( x \neq 2 \)

\[ \therefore x = 5 \]

6. Let \( \left| \frac{1 - z_1 z_2}{z_1 - z_2} \right| < 1 \)

\[ \Rightarrow |1 - z_1 z_2| < |z_2 - z_1| \]

\[ \Rightarrow (1 - z_1 z_2)(1 - z_1 z_2) < (z_2 - z_1)(z_2 - z_1) \]

\[ \Rightarrow 1 + |z_1|^2 |z_2|^2 - |z_1|^2|z_2|^2 < 0 \]

\[ \Rightarrow (1 - |z_1|^2)(1 - |z_2|^2) < 0 \]

which is true because of \( |z_1| < 1 < |z_2| \).

7. \( (2x)^{m_2} = (3y)^{m_3} \)

\[ \Rightarrow \ell n 2 \cdot \ell n (2x) = \ell n 3 \cdot \ell n (3y) = \ell n 3 (\ell n 3 + \ell n y) \]

\[ \Rightarrow 3^{\ell n x} = 2^{\ell n y} \]

also \( 3^{\ell n x} = 2^{\ell n y} \)

by (1) \[ \Rightarrow \ell n 2 \cdot \ell n (2x) = \ell n 3 (\ell n 3 + \ell n y) \]

\[ \Rightarrow \ell n 2 \cdot \ell n 2x = \ell n 2 (\ell n 2 + \ell n x) \]

\[ \Rightarrow \ell n 2x = 0 \]

\[ \Rightarrow x = \frac{1}{2} \]

8. Let \( \sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} = t \)

\[ \sqrt{4 - \frac{1}{3\sqrt{2}}} t = t \]

\[ 4 - \frac{1}{3\sqrt{2}} t = t^2 \]

\[ t^2 + \frac{1}{3\sqrt{2}} t - 4 = 0 \]

\[ \Rightarrow 3\sqrt{2} t^2 + t - 12\sqrt{2} = 0 \]

\[ t = \frac{-1 \pm \sqrt{1^2 - 4 \times 3\sqrt{2} \times 12\sqrt{2}}}{2 \times 3\sqrt{2}} = \frac{-1 \pm 17}{2 \times 3\sqrt{2}} \]

\[ t = \frac{16}{6\sqrt{2}}, \frac{-18}{6\sqrt{2}} \]
t = \frac{8}{3\sqrt{2}}, \frac{-3}{\sqrt{2}} and \frac{-3}{\sqrt{2}} is rejected

so 6 + \log_{22} \left( \frac{1}{3\sqrt{2}} \times \frac{8}{3\sqrt{2}} \right) = 6 + \log_{22} \left( \frac{4}{9} \right)

= 6 + \log_{22} \left( \frac{2}{3} \right)

= 6 - 2 = 4

\textbf{PART - II}

1. Let \( z = r_1 e^{i\theta} \) and \( w = r_2 e^{i\phi} \)

\( \Rightarrow \bar{z} = r_1 e^{-i\theta} \)

Given, \( |z_\omega| = 1 \)

\( \Rightarrow r_1 r_2 = 1 \) \hspace{1cm} ... (i)

and \( \arg (z) - \arg (\omega) = \frac{\pi}{2} \)

\( \Rightarrow 0 - \phi = \frac{\pi}{2} \)

Then, \( \bar{z} \omega = r_1 e^{-i\theta} \cdot r_2 e^{i\phi} \)

\( = r_1 r_2 e^{i(\theta - \phi)} \)

From Eqs. (i) and (ii), we get

\( \bar{z} \omega = 1. e^{-i\pi/2} \)

\( = \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \)

\( \Rightarrow \bar{z} \omega = -i \).

2. \( \left( \frac{1+i}{1-i} \right)^x = \left( \frac{(1+i)(1+i)}{(1-i)(1+i)} \right)^x = \left( \frac{(1+i)^2}{1-i^2} \right)^x = \left[ \frac{1-1+2i}{2} \right]^x \)

\( \Rightarrow \left( \frac{1+i}{1-i} \right)^x = (i)^x = 1 \) \hspace{1cm} \text{given}

\( \Rightarrow (i)^x = (i)^n \),

where \( n \) is any positive integer.

\( \Rightarrow x = 4n \).

3. Since \( \bar{z} + i \bar{w} = 0 \) \( \Rightarrow \bar{z} = -i \bar{w} \)

\( \Rightarrow z = iw \) \( \Rightarrow w = -iz \)

Also, \( \arg(zw) = \pi \)

\( \Rightarrow \arg(-iz^2) = \pi \)

\( \Rightarrow \arg(-i) + 2 \arg(z) = \pi \)

\( \Rightarrow -\frac{\pi}{2} + 2 \arg(z) = \pi \)

\( \therefore \arg(-i) = -\frac{\pi}{2} \)

\( \Rightarrow 2 \arg(z) = \frac{3\pi}{2} \)

\( \Rightarrow \arg(z) = \frac{3\pi}{4} \).

4. Let \( z = \frac{1}{i-1} \)

\( \therefore \bar{z} = \left[ \frac{1}{i-1} \right] = \frac{1}{-i-1} = -\frac{1}{i+1} \).

5. Let roots be \( p + iq \) and \( p - iq \) \( \quad p, q \in \mathbb{R} \)

root lie on line \( \text{Re}(z) = 1 \)

\( \Rightarrow p = 1 \)

product of roots = \( p^2 + q^2 = \beta = 1 + q^2 \)

\( \Rightarrow \beta \in (1, \infty), \quad (q \neq 0, \quad \therefore \text{roots are distinct}) \quad \text{Ans.} \)
Section (A):

A-3._ (i) centroid \( \left( \frac{0 + 5 + 16}{3}, \frac{0 + 12 + 12}{3} \right) = (7, 8) \)

(ii) Let coordinates of circumcentre is O (x, y).
Therefore \( OA = OB = OC \)
\[
\Rightarrow x^2 + y^2 = (x - 5)^2 + (y - 12)^2 = (x - 16)^2 + (y - 12)^2
\]
\[
\Rightarrow x^2 + y^2 = (x - 5)^2 + (y - 12)^2 \Rightarrow 10x + 24y = 16 \quad g(x - 5)^2 + (y - 12)^2 = (x - 16)^2 + (y - 12)^2
\]
\[
\Rightarrow 2x = 21 \Rightarrow x = \frac{21}{2}, y = \frac{8}{3}
\]

(iii) \( I = \left( \frac{0 \times 11 + 5 \times 20 + 16 \times 13}{13 + 20 + 11}, \frac{0 \times 11 + 12 \times 20 + 13 \times 12}{13 + 20 + 11} \right) = (7, 9) \)

(iv) \( I_2 = \left( \frac{-5 \times 20 + 13 \times 16 + 11 \times 13}{-20 + 13 + 11}, \frac{-12 \times 20 + 0 \times 11 + 13 \times 12}{-20 + 13 + 11} \right) = (27, -21) \)

A-4. Let coordinates of P(x,y)
given PA = PB
\[
\Rightarrow (x - 3)^2 + (y - 4)^2 = (x - 5)^2 + (y + 2)^2
\]
\[
\Rightarrow 4x - 12y = 4
\]
\[
\Rightarrow x - 3y = 1 \quad \ldots (i)
\]
\[
\begin{vmatrix}
1 & x & 1 \\
3 & y & 1 \\
2 & 5 & 1
\end{vmatrix} = 10
\]
\[
\Rightarrow 6x + 2y - 26 = \pm 20 \quad \Rightarrow 3x + y - 13 = \pm 10
\]
\[
\Rightarrow 3x + y = 23 \quad \ldots (ii)
\]
\[
\Rightarrow 3x + y = 3 \quad \ldots (iii)
\]
Solving (i) and (ii) we get (7, 2)
Solving (i) and (iii) we get (1, 0)

Section (B):

B-2. Let equation of line is \( lx + my + n = 0 \) \ldots (i)
given \( \left( \frac{a^2}{a - 1}, \frac{a^2 - 3}{a - 1} \right), \left( \frac{b^3}{b - 1}, \frac{b^2 - 3}{b - 1} \right) \) and \( \left( \frac{c^3}{c - 1}, \frac{c^2 - 3}{c - 1} \right) \) are collinear

\[
\left( \frac{t^3}{t - 1}, \frac{t^2 - 3}{t - 1} \right) \text{ is general point which satisfies line (i)}
\]
\[
\Rightarrow \ell \left( \frac{t^3}{t - 1} \right) + m \left( \frac{t^2 - 3}{t - 1} \right) + n = 0
\]
\[
\Rightarrow \ell t^3 + m t^2 + nt - (3m + n) = 0
\]
\[a + b + c = \frac{m}{\ell}\]
\[ab + bc + ac = \frac{n}{\ell}\]
\[abc = \frac{3m + n}{\ell}\]

Now
\[\text{LHS} = abc - (ab + bc + ac) + 3(a + b + c)\]
\[= \left(\frac{3m + n}{\ell}\right) - \frac{n}{\ell} + 3 \left(\frac{-m}{\ell}\right) = 0\]

**B-5.** Let point is \(P(x, y)\) and \(A(\alpha e, 0)\) and \(B(-\alpha e, 0)\)

Given \(|PA - PB| = 2a\) \(\Rightarrow\) \[\sqrt{(x - \alpha e)^2 + y^2} - \sqrt{(x + \alpha e)^2 + y^2} = 2a\]

Let \[\sqrt{(x - \alpha e)^2 + y^2} = A, \sqrt{(x + \alpha e)^2 + y^2} = B\]

Hence \[A - B = 2a\]

\[A^2 - B^2 = (A + B)(A - B)\]
\[\Rightarrow A + B = -2\alpha e\]

\[A = \alpha - \alpha e\]

\[(x - \alpha e)^2 + y^2 = (\alpha - \alpha e)^2\]
\[\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1\]

**Section (C):**

**C-3.** Obvious

**C-4.** By parametric form

\[Q \left(4 + \frac{11}{2\sqrt{2}} \cos \theta, 1 + \frac{11}{2\sqrt{2}} \sin \theta\right)\]

it lies on \(3x - y = 0\)
\[\Rightarrow 12 + \frac{33}{2\sqrt{2}} \cos \theta - 1 - \frac{11}{2\sqrt{2}} \sin \theta = 0\]
\[\Rightarrow 1 + \frac{3}{2\sqrt{2}} \cos \theta - \frac{\sin \theta}{2\sqrt{2}} = 0\]
\[\Rightarrow 3\cos \theta - \sin \theta = -2\sqrt{2}\]

squaring both sides
\[9\cos^2 \theta + \sin^2 \theta - 6\sin \theta \cos \theta = 8(\sin^2 \theta + \cos^2 \theta)\]
\[\cos^2 \theta - 6\sin \theta \cos \theta - 7\sin^2 \theta = 0\]

\[7\tan^2 \theta + 6\tan \theta - 1 = 0\]
\[\tan \theta = -1, \frac{1}{7}\]

Hence required line are \(x + y = 5, x - 7y + 3 = 0\)

**Section (D):**

**D-2.** foot of perpendicular

\[\frac{x - 2}{3} = \frac{y - 3}{-1} = \frac{(3 \times 2 - 3 - 4)}{3^2 + (-1)^2} \Rightarrow (x, y) = \left(\frac{23}{10}, \frac{29}{10}\right)\]

image

\[\frac{x - 2}{3} = \frac{y - 3}{-1} = -2 \frac{(3 \times 2 - 3 - 4)}{3^2 + (-1)^2} \Rightarrow (x, y) = \left(\frac{13}{5}, \frac{14}{5}\right)\]
slop of line perpendicular to the line \( y = 3x - 4 \) is \(- \frac{1}{3}\) hence its equation

\[ y - 3 = -\frac{1}{3} (x - 2) \quad \Rightarrow \quad x + 3y - 11 = 0 \]

D-5. \( L_1 : 4x + 3y - 7 = 0 \)
\( L_2 : 24x + 7y - 31 = 0 \)
\( a_1, a_2 + b_1, b_2 = 4 \times 24 + 3 \times 7 > 0 \)
Hence + sign gives obtuse angle bisector and – sign gives acute angle bisector
Now, put origin in both \( 4 \times 0 + 3 \times 0 - 7 < 0 \)
\( 24 \times 0 + 7(0) - 31 < 0 \)
Hence + sign gives that bisector in which origin lies.
Hence origin lies in obtuse angle bisector
Now, equation of bisector \( \left( \frac{4x + 3y - 7}{5} \right) = \pm \left( \frac{24x + 7y - 31}{25} \right) \)
\[ \Rightarrow \quad + \text{sign} \quad x - 2y + 1 = 0 \]
\[ -\text{sign} \quad 2x + y - 3 = 0 \]

Section (E) :

E-4. \( 12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0 \)
This represents pair of straight lines if \( \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \)
we get \( \lambda = 2 \)
Now
\( 12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = (6x - 2y + p) (2x - y + q) \)
compair both sides \( 2p + 6q = 11 \)
\( -p - 2q = -5 \)
solving both we get \( p = 4, q = \frac{1}{2} \)
Hence required lines are \( 6x - 2y + 4 = 0 \Rightarrow 3x - y + 2 = 0 \)
\[ 2x - y + \frac{1}{2} = 0 \Rightarrow 4x - 2y + 1 = 0 \]

solving both equations we get point of intersection \( \left( \frac{-3}{2}, \frac{-5}{2} \right) \)

Now angle between both lines
\[ \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1m_2} \right| = \left| \frac{3 - 2}{1 + 3 \times 2} \right| = \frac{1}{7} \Rightarrow \theta = \tan^{-1} \frac{1}{7} \]

Now equation of pair of angle bisector
\[ \left( \frac{x + 3}{2} \right)^2 - \left( \frac{y + 5}{2} \right)^2 = \left( \frac{x + 3}{2} \right)\left( \frac{y + 5}{2} \right) \quad \Rightarrow \quad 2x^2 + 4xy - 2y^2 + 16x - 4y + 7 = 0 \]

E-5. Homogenize \( x^2 + y^2 = a^2 \) by \( y = mx + c \)
we get \( x^2 + y^2 = a^2 \left( \frac{y - mx}{c} \right)^2 \)
This equation represents pair of lines passing through origin.
That will be right angle if
coeff. \( x^2 + \text{coeff.} \ y^2 = 0 \Rightarrow 2c^2 = a^2 (1 + m^2) \)

Section (F) :

F-1. (i) (2, 5, 8) (ii) (–5, –4, –3) (iii) (–3, 0, 7) (iv) (8, 2, 5)
PART - II

Section (A) :

A-1*: \( AB = \sqrt{4 + 9} = \sqrt{13} \)  
\( BC = \sqrt{36 + 16} = 2\sqrt{13} \)  
\( CD = \sqrt{4 + 9} = \sqrt{13} \)  
\( AD = \sqrt{36 + 16} = 2\sqrt{13} \)  
\( AC = \sqrt{64 + 1} = \sqrt{65} \)  
\( BD = \sqrt{16 + 49} = \sqrt{65} \)

If H is orthocentre of triangle ABC, then orthocentre of triangle BCH is point A

Section (B) :

B-2*: Since A, B, C are collinear, near  
Slope of AB = Slope of BC  
\( \frac{2 - 2k - 2k}{k - 1 + k} = \frac{2k - 6 - 2k}{1 - k + k + 4} \)  
\( \Rightarrow \frac{2 - 4k}{2k - 1} = \frac{4k - 6}{5} \)  
\( \Rightarrow 10 - 20k = (4k - 6)(2k - 1) \)  
\( \Rightarrow (4k - 6)(2k - 1) + 10(2k - 1) = 0 \)  
\( \Rightarrow k = \frac{1}{2}, -1 \)
B-3. \[ \text{AP} = \sqrt{x^2 + (y - 4)^2} \]
\[ \text{BP} = \sqrt{x^2 + (y + 4)^2} \]
\[ \text{AP} - \text{BP} = \pm 6 \]
\[ \sqrt{x^2 + (y - 4)^2} - \sqrt{x^2 + (y + 4)^2} = \pm 6 \]
On squaring we get the locus of P
\[ 9x^2 - 7y^2 + 63 = 0 \]

Section (C) :

C-2. \[ x_1 + y_1 = 5 \] \hspace{1cm} ... (i)
\[ x_2 = 4 \] \hspace{1cm} ... (ii)
co - ordinates of G are \( (4, 1) \)
\[ \Rightarrow \frac{1 + x_1 + x_2}{3} = 4 \] \hspace{1cm} ...(iii)
and \((एवा)\) \[ \frac{y_1 + y_2 + 2}{3} = 1 \] \hspace{1cm} ...(iv)
solving above equations, we get B & C.

C-4. Let coordinates of point P by parametric
\[ P(2 + r \cos 45^{\circ}, 3 + r \sin 45^{\circ}) \]
It satisfies the line \( 2x - 3y + 9 = 0 \)
\[ 2 \left( 2 + \frac{r}{\sqrt{2}} \right) - 3 \left( 3 + \frac{r}{\sqrt{2}} \right) + 9 = 0 \]
\[ \Rightarrow r = 4\sqrt{2} \]

Section (D) :

D-1. \[ a^2x + ab + 1 = 0 \]
origin and \((1, 1)\) lies on same side.
\[ a^2 + ab + 1 > 0 \quad \forall \ a \in R \]
\[ D < 0 \quad \Rightarrow \quad b^2 - 4 < 0 \]
\[ \Rightarrow \quad b \in (-2, 2) \]
but \[ b > 0 \quad \Rightarrow \quad b \in (0, 2) \]

D-4. \[ p = \left| \frac{-22 - 64 - 5}{2^2 + (-16)^2} \right| = \frac{91}{260} \]
\[ q = \left| \frac{-64 \times 11 + 8 \times 4 + 35}{64^2 + 8^2} \right| \]
\[ p < q \quad \text{Hence} \ 2x - 16y - 5 = - \text{is acute angle bisector} \]
Section (E) :

E-2. \( m_1 + m_2 = -10 \)
\[
m_1m_2 = \frac{a}{1}
\]
given \( m_1 = 4m_2 \) \( \Rightarrow \) \( m_2 = -2, m_1 = -8 \),
a = 16

E-5. Homogenize given curve with given line
\[
3x^2 + 4xy - 4x(2x + y) + 1(2x + y)^2 = 0
\]
\[
3x^2 + 4xy - 8x^2 - 4xy + 4x^2 + y^2 + 4xy = 0
\]
\[
- x^2 + 4xy + y^2 =
\]
coeff. \( x^2 + \text{coeff. } y^2 = 0\)
Hence angle is 90°

Section (F) :

F-3. \( x^2 + y^2 + z^2 + x^2 + y^2 = 36 \)
\[
2(x^2 + y^2 + z^2) = 36
\]
\[
\sqrt{x^2 + y^2 + z^2} = 3\sqrt{2}
\]

F-4. The two numbers are \( x \) and \( x + 2 \)
(a) \( x > 10 \)
(b) \( x + 2 > 10 \) \( \Rightarrow \) \( x > 8 \)
(c) \( x + x + 2 < 34 \)
\[
2x < 32 \Rightarrow x < 16
\]
Now \( x \) must be between \( 10 < x < 16 \)
\( x \in (11, 13), (13, 15) \)

F-6. Let the third PH reading is \( x \)
\[
7.4 < \frac{7.48 + 8.42 + x}{3} < 8.2
\]
\[
22.2 < 15.90 + x < 24.6
\]
\[
6.3 < x < 8.7
\]
PH range should be in between 6.3 to 8.7

F-8. Standard result.

EXERCISE # 2

PART - I

3. A, S, B are collinear
\[
\begin{bmatrix}
0 & 0 & 1 \\
x_1 & x_2 - x_1 & 1 \\
2 & 1 & 1
\end{bmatrix}
= 0
\]
\[ 3x_1 = 2x_2 \quad \ldots \quad (1) \]

B, R, C are collinear

\[
\begin{vmatrix}
2 & 1 & 1 \\
x_2 & x_2 - x_1 & 1 \\
3 & 0 & 1 \\
\end{vmatrix} = 0 \quad \Rightarrow \quad x_1 - 2x_2 + 3 = 0 \quad \ldots \quad (2)
\]

Solving (i) and (ii) we get

\[ x_1 = \frac{3}{2} \]

\[ x_2 = \frac{9}{4} \]

Hence

\[ P \left( \frac{3}{2}, 0 \right), Q \left( \frac{9}{4}, 0 \right), \quad R \left( \frac{3}{2}, \frac{3}{4} \right), \quad S \left( \frac{9}{4}, \frac{3}{4} \right) \]

6.

(i) D is mid point of BC

Hence co-ordinates of D are \( \left( \frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2} \right) \)

Therefore, equation of the median AD is

\[
\begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
\frac{x_2 + x_3}{2} & \frac{y_2 + y_3}{2} & 1 \\
\end{vmatrix} = 0
\]

Applying \( R_3 \rightarrow 2R_3 \)

\[
\begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_2 + x_3 & y_2 + y_3 & 2 \\
\end{vmatrix} = 0
\]

\[
\begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
\end{vmatrix} + \begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_3 & y_3 & 1 \\
\end{vmatrix} = 0
\]

(Using the addition property of determinants)

(ii) Let \( P(x, y) \) be any point on the line parallel to BC

Area of \( \triangle ABP = \) Area of \( \triangle ACP \)

\[
\begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
\end{vmatrix} = \begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_3 & y_3 & 1 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
\end{vmatrix} - \begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_3 & y_3 & 1 \\
\end{vmatrix} = 0
\]

This gives the equation of line AP.

(iii) Let AD be the internal bisector of angle A,

\[
\frac{BD}{DC} = \frac{BA}{CA} = \frac{c}{b}
\]

\[
D = \left( \frac{cx_3 + bx_2}{c + b}, \frac{cy_3 + by_2}{c + b} \right)
\]
Let \( P(x, y) \) be any point on \( AD \) then \( P, A, D \) are collinear

\[
\begin{vmatrix}
 x & y & 1 \\
 x_1 & y_1 & 1 \\
 cx_3 + bx_2 & cy_3 + by_2 & 1 \\
 b + c & b + c & 1 \\
\end{vmatrix} = 0
\]

\( R_3 \rightarrow (b + c) R_3 \)

\[
\begin{vmatrix}
 x & y & 1 \\
 x_1 & y_1 & 1 \\
 cx_3 + bx_2 & cy_3 + by_2 & 1 \\
 b & b & c \\
\end{vmatrix} = 0
\]

\( \therefore \)

\[
\begin{vmatrix}
 x & y & 1 \\
 x_1 & y_1 & 1 \\
 cx_3 & cy_3 & 1 \\
 bx_2 & by_2 & b \\
\end{vmatrix} + \begin{vmatrix}
 x & y & 1 \\
 x_1 & y_1 & 1 \\
 bx_2 & by_2 & b \\
 x_2 & y_2 & 1 \\
\end{vmatrix} = 0
\]

(Addtion property)

\[
\Rightarrow c \begin{vmatrix}
 x & y & 1 \\
 x_1 & y_1 & 1 \\
 x_3 & y_3 & 1 \\
\end{vmatrix} + b \begin{vmatrix}
 x & y & 1 \\
 x_1 & y_1 & 1 \\
 x_2 & y_2 & 1 \\
\end{vmatrix} = 0
\]

This is the equation of \( AD \).

9. equation of line \( L_1 \) is

\[
y - \frac{5}{2} = 2 \cdot (x - \frac{3}{2})
\]

or \( 2x - y - \frac{1}{2} = 0 \)

or \( 4x - 2y - 1 = 0 \)

equation of line \( L_2 \) is

\[
y - \frac{5}{2} = 1 \cdot (x - \frac{3}{2}) \quad \text{or} \quad x - y + 1 = 0
\]

Point \( C \) is mirror image of point \( A \) w.r.t line \( L_1 \)

\[
\frac{x - (-2)}{4} = \frac{y - (3)}{-2} = \frac{-2(-8 - 6 - 1)}{20}
\]

\[
\therefore \ C(4, 0)
\]

similarly \( B \) is mirror image of \( A \) in line \( L_2 = 0 \)

\[
\frac{x - (-2)}{1} = \frac{(y - 3)}{-1} = \frac{-2(-2 - 3 + 1)}{2}
\]

\[
\therefore \ B(2, -1) \Rightarrow D(1, \frac{3}{2}) ; E(0, 1)
\]

median through \( B \) is

\[
(y + 1) = \frac{5/2}{-1} (x - 2) \Rightarrow 5x + 2y = 8
\]

median through \( C \) is

\[
(y - 1) = \frac{1}{-4}(x - 0) \Rightarrow x + 4y = 4
\]

11. \( a^2 + b^2 = c^2 \)

\[\text{.... (i)}\]

Let \( L \) is \((x_1, y_1) \)

\( L \) is foot of perpendicular from point \( P(a, b) \) on line \( AB \)

equation of \( AB \) is \( bx + ay - ab = 0 \)

\[
\Rightarrow \frac{x_1 - a}{b} = \frac{y_1 - b}{a} = -\frac{ab + ab - ab}{a^2 + b^2}
\]

\[\text{Solutions (XI) # 46}\]
\[
\frac{x_1 - a}{b} = \frac{y_1 - b}{a} = -\frac{ab}{c^2}
\]

\[\Rightarrow x_1 = a - \frac{ab^2}{c^2} = \frac{a(c^2 - b^2)}{c^2} = a^3/c^2 \]

\[\Rightarrow a^3 = c^2x_1 \quad \text{.... (ii)} \]

similarly

\[b^3 = c^2y_1 \quad \text{.... (iii)} \]

using these relations (ii) & (iii) in equation (i), we get required locus.

14. Given pair of lines are \(a^2x^2 + 2h(a + b)xy + b^2y^2 = 0\)

\[\Rightarrow ax^2 + 2hxy + by^2 = 0 \]

Equation of pair of bisectors of first pair is

\[
\frac{x^2 - y^2}{a^2 - b^2} = \frac{xy}{h(a + b)}
\]

\[\Rightarrow \frac{x^2 - y^2}{a-b} = \frac{xy}{h} \]

Which is also bisector of second pair. Hence both pair are equally inclined.

15. Let equation of chords \(hx + ky = 1\)

By homogenisation

\[3x^2 - y^2 - 2x(hx + ky) + 4y(hx + ky) = 0 \]

\[\Rightarrow \text{it makes } 90^\circ. \text{ Hence} \]

\[\text{coeff. } x^2 \text{ & coeff. } y^2 = 0 \]

\[3 - 2h - 1 + 4k = 0 \Rightarrow h - 2k = 1 \]

\[\text{Hence all chords are concurrent at } (1, -2) \]

Similarly homogenize \(3x^2 - y^2 - 2x + 4y = 0\)

\[\Rightarrow 3x^2 + 3y^2 - 2x(hx + ky) + 4y(hx + ky) = 0 \]

\[\text{again coeff. } x^2 \text{ & coeff. } y^2 = 0 \]

\[3 + 3 - 2h + 4k = 0 \Rightarrow h - 2k = 3 \Rightarrow \frac{h}{3} - \frac{2k}{3} = 1 \]

Hence, all chords passes through \(\left(\frac{1}{3}, \frac{-2}{3}\right)\).

**PART - II**

1. here \(\tan \theta = \frac{1}{5}\)

\[\therefore \quad \tan 2 \theta = \frac{2\left(\frac{1}{5}\right)}{1 - \left(\frac{1}{5}\right)^2} = \frac{5}{12} \]

\[\therefore \quad \text{required line } y = \frac{5x}{12} \]

4. \(p = \left|\frac{0 + 0 - a}{\sqrt{5}}\right| = \frac{a}{\sqrt{5}}\)

\[\tan 45^\circ = \frac{p}{x} \Rightarrow p = x \]

\[\text{Hence area } = \frac{1}{2} (2x)(p) = px = p^2 = a/5 \]
8. Image of A(3, 10) in \(2x + y - 6 = 0\)
\[
\frac{x - 3}{2} = \frac{y - 10}{1} = -2 \left(\frac{6 + 10 - 6}{2^2 + 1^2}\right)
\]
A' = \((-5, 6)\)

Equation of A'B is \(y - 3 = \left(\frac{6 - 3}{-5 - 4}\right)(x - 4)\)
\[
y - 3 = -\frac{1}{3}(x - 4)
\]
\[
3y - 9 = -x + 4
\]
\[
\Rightarrow x + 3y - 13 = 0
\]

10. By geometry
\[a^2 + b^2 = (a + b)^2 \quad \text{....(i)}\]

By section formula
\[
h = \frac{\alpha}{a+b} \quad \Rightarrow \quad \alpha = \frac{n(a+b)}{a}
\]
\[
k = \frac{\beta}{a+b} \quad \Rightarrow \quad \beta = \frac{k(a+b)}{b}
\]

Put value of \(\alpha\) and \(\beta\) in (i)
\[
\frac{h^2(a+b)^2}{a^2} + \frac{k^2(a+b)^2}{b^2} = (a + b)^2
\]
\[
\Rightarrow \quad \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1
\]

Locus of (opchim) is \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\)

12. \(x^2 - 2pxy - y^2 = 0\)

pair of angle bisector of this pair \(\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p}\)
\[
\Rightarrow \quad x^2 - y^2 + \frac{2}{p} xy = 0
\]

compare this bisector pair with \(x^2 - 2qxy - y^2 = 0\)
\[
\frac{2}{p} = -2q \Rightarrow pq = -1.
\]

14. \(\angle 1 = 2x - 3y - 6 = 0\)
\(\angle 2 = 3x - y + 3 = 0\)
\(\angle 3 = 3x + 4y - 12 = 0\)

Hence \(\alpha \in [-1, 3]\)
\(\beta \in [-2, 3]\)
16. Both A & B are same side of line $2x - 3y - 9 = 0$
Now, perimeter of $\triangle APB$ will be least when points A, P, B were collinear. Let $B'$ is image of B
\[
\frac{x - 0}{2} = \frac{y - 4}{-3} = -2 \left( \frac{0 - 12 - 9}{2^2 + (-3)^2} \right) \quad \text{A}(-2,0) \quad \text{B}(0, 4) \quad \text{P} \quad \text{2x} - 3y - 9 = 0 \\
\Rightarrow \quad B' \left( \frac{84}{13} , -\frac{74}{13} \right) \\
\]
Now equation of $AB'$ is $y = \frac{-74}{110} (x + 2)$
point of intersection of given line & Q is $P \left( \frac{21 - 37}{17} , \frac{-37}{17} \right)$.

EXERCISE # 3

1. (A) Slope of such line is $\pm 1$

(B) area of $\triangle OAB = \frac{1}{2} \times 3 \times 4 = 6$ sq. units

(C) To represent pair of straight lines
\[
\begin{vmatrix} 2 & -1 & -3 \\ -1 & 1 & 3 \\ -3 & 3 & c \end{vmatrix} = 0 \quad \Rightarrow \quad c = 3
\]

(D) Lines represented by given equation are $x + y + a = 0$ and $x + y - 9a = 0$
\[\therefore \text{distance between these parallel lines is } \frac{10a}{\sqrt{2}} = 5\sqrt{2}a\]

Comprehension # 2 (5, 6, 7)
Slopes of the lines

$3x + 4y = 5$ is $m_1 = -\frac{3}{4}$

and $4x - 3y = 15$ is $m_2 = \frac{4}{3}$
\[\therefore \text{given lines are perpendicular and } \angle A = \frac{\pi}{2}\]

Now required equation of BC is
\[
(y - 2) = \frac{m \pm \tan(\pi/4)}{1 \mp m \tan(\pi/4)} (x - 1) \quad \text{......(1)}
\]

where $m = \text{slope of AB} = -\frac{3}{4}$
\[\therefore \text{equation of BC is (on solving (1))} \quad x - 7y + 13 = 0 \text{ and } 7x + y - 9 = 0 \]
$L_1 = x - 7y + 13 = 0$
$L_2 = 7x + y - 9 = 0$

5. $c + f = 4$

6. Equation of a straight line through (2, 3) and inclined at an angle of $(\pi/3)$ with y-axis ($(\pi/6)$ with x-axis) is
\[
\frac{x - 2}{\cos(\pi/6)} = \frac{y - 3}{\sin(\pi/6)} \quad \Rightarrow \quad x - \sqrt{3}y = 2 - 3\sqrt{3}
\]
Points at a distance $c + f = 4$ units from point P are
\[
(2 + 4 \cos (\pi/6), 3 + 4 \sin (\pi/6)) = (2 + 2\sqrt{3}, 5)\]
and \((2 - 4 \cos (\pi/6), 3 - 4 \sin (\pi/6)) = (2 - 2\sqrt{3}, 1)\)

only (A) is true out of given options

7. Let required line be \(x + y = a\)
   which is at \(|b - 2a - 1| = |5 - 4 - 4\sqrt{3} - 1| = 4\sqrt{3}\) units from origin
   \[\therefore \text{required line is } x + y - 4\sqrt{6} = 0 \text{ (since intercepts are on positive axes only)}\]

8. \(ax^3 + bx^2y + cxy^2 + dy^3 = 0\)
   since this is homogeneous pair represent there straight lines passing through origin
   \[ax^3 + bx^2y + cxy^2 + dy^3 = (y - m_1x)(y - m_2x)(y - m_3x)\]
   or put \(y = mx\) in given equation we get
   \[m_1 + m_2 + m_3 = \frac{-c}{d}\]
   \[m_1m_2 + m_2m_3 + m_3m_1 = \frac{b}{d}\]
   \[m_1m_2m_3 = \frac{-a}{d}\]
   given two lines + hence \(m_1, m_2 = -1 \Rightarrow m_3 = a/d\)
   eliminate \(m_3\) from remaining equation

10. \(S_2\) is standard result.
    equation of angle bisectors of lines given in \(S_1\) are
    \[
    \frac{3x + 4y + 2}{5} = \pm \frac{4x + 3y - 2}{5} \Rightarrow x - y = 0 \text{ and } 7x + 7y - 24 = 0
    \]

14._ Let \(R(5, 1)\) divides line segment joining \(P(2,10)\) and \(Q(6, -2)\) in \(\lambda : 1\)
    \[
    \frac{5}{1} = \frac{6\lambda + 2}{\lambda + 1} \Rightarrow \lambda = 3
    \]
    Hence Harmonic conjugate divides in \(3 : 1\) externally
    Hence required part is \(\left( \frac{18 - 2}{3 - 1}, \frac{-6 - 10}{3 - 1} \right) = (8, -8)\)

19. Required point is foot of perpendicular from \((0, 0)\) on the given line which is
    \[
    \frac{\alpha - 0}{3} = \frac{\beta - 0}{4} = \frac{-(-1)}{25}
    \]

**EXERCISE # 4**

**PART - I**

1. \(A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)\)
   \(x_1, x_2, x_3\) and \(y_1, y_2, y_3\) are in G.P. of common ratio \(r\).
   \[\therefore x_2 = x_1r, x_3 = x_1r^2, y_2 = y_1r, y_3 = y_1r^2\]
   Area of triangle \(ABC = \frac{1}{2} \begin{vmatrix} x_1 & x_1r & x_1r^2 \\ y_1 & y_1r & y_1r^2 \\ 1 & 1 & 1 \end{vmatrix} = 0 \therefore A, B & C\text{ are collinear.}\]

2. Let \(m\) be the slope of \(PQ\) then
   \[
   \tan 45^\circ = \left| \frac{m - (-2)}{1 + m(-2)} \right|
   \Rightarrow 1 = \left| \frac{m + 2}{1 - 2m} \right|
   \]
   \[
   R
   \]

**Solutions (XI) # 50**
\[ \pm 1 = \frac{m + 2}{1 - 2m} \]
\[ m + 2 = 1 - 2m \text{ or } -1 + 2m = m + 2 \]
\[ m = -\frac{1}{3} \text{ or } m = 3 \]

PR makes 45° with PQ

equation of PQ \ y - 1 = -\frac{1}{3} (x - 2)
\[ \Rightarrow \ x + 3y - 5 = 0 \]
equation of PR is \ y - 1 = 3(x - 2)
\[ \Rightarrow \ 3x - y - 5 = 0 \]
combined equation of PQ and PR (x + 3y - 5) (3x - y - 5) = 0
\[ \Rightarrow \ 3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0 \]

3. S is the mid point of Q and R
\[ S \left( \frac{7 + 6}{2}, \frac{3 - 1}{2} \right) = \left( \frac{13}{2}, -1 \right) \]
slope of PS = \[ m = \frac{2 - 1}{2 - -13}/2 = \frac{-2}{9} \]
equation of line passing through (1, -1) and parallel to PS is
\[ y + 1 = \frac{-2}{9} (x - 1) \quad \Rightarrow \quad 2x + 9y + 7 = 0 \]

4.

BC = 2
AB = 2
AC = 2
Hence ABC is an equilateral triangle. In equilateral triangle incentre coincides with centroid. Thus
\[ I = \left( \frac{0 + 2 + 1}{3}, \frac{0 + 0 + \sqrt{3}}{3} \right) = \left( 1, \frac{1}{\sqrt{3}} \right) \]

5. \( p (h, k) \) be a general point in the first quadrent such that \( d(P, A) = d(p, O) \)
\[ \Rightarrow \ |h - 3| + |k - 2| = |h| + |k| = h + k \]
[h and k are positive point (h, k) being in I quadrant]
If \( h < 3, k < 2 \), then (h, k) lie in region I
If \( h > 3, k < 2 \), then (h, k) lie in region II
If \( h > 3, k > 2 \), then (h, k) lie in III
If \( h < 3, k > 2 \), then (h, k) lie in IV

In region I \( 3 - h + 2 - k = h + k \) \[ \Rightarrow \quad h + k = \frac{5}{2} \]
In region II \( h - 3 + 2 - k = h + k \) \[ \Rightarrow \quad k = -\frac{1}{2} \text{ not possible} \]
In region III \( h - 3 + k - 2 = h + k \quad \Rightarrow \quad -5 = 0 \) Not possible

In region IV \( 3 - h + k - 2 = h + k \Rightarrow \quad h = \frac{1}{2} \)

Set consist line segment \( x + y = \frac{5}{2} \) of finite length

In 1st region and \( x = \frac{1}{2} \) in the IV region.

6.
\[
c_1 \rightarrow ac_1
\]
\[
= \frac{1}{a} \begin{vmatrix}
  a^2x - aby - ac & bx + ay & cx + a \\
  abx + a^2y & -ax + by - c & cy + b \\
  acx + a^2 & cy + b & -ax - by + c
\end{vmatrix}
\]
\[
c_1 \rightarrow c_1 + bc_2 + cc_3
\]
\[
\Delta = \frac{1}{a} \begin{vmatrix}
  (a^2 + b^2 + c^2)x & ay + bx & cx + a \\
  (a^2 + b^2 + c^2)y & by - c - ax & cy + b \\
  (a^2 + b^2 + c^2) & b + cy & -ax - by + c
\end{vmatrix}
\]
As \( a^2 + b^2 + c^2 = 1 \)
\[
c_2 \rightarrow c_2 - bc_1, \quad c_3 \rightarrow c_3 - cc_1
\]
\[
\Delta = \frac{1}{a} \begin{vmatrix}
  x & ay & a \\
  y & c - ax & b \\
  1 & cy & -ax - by
\end{vmatrix}
\]
\[
R_1 \rightarrow x R_1
\]
\[
= \frac{1}{ax} \begin{vmatrix}
  x^2 & ax & ax \\
  y & c - ax & b \\
  1 & cy & -ax - by
\end{vmatrix}
\]
\[
R_1 \rightarrow R_1 + yR_2 + R_3
\]
\[
\Delta = \frac{1}{ax} \begin{vmatrix}
  x^2 + y^2 + 1 & 0 & 0 \\
  y & c - ax & b \\
  1 & cy & -ax - by
\end{vmatrix}
\]
\[
\Rightarrow \Delta = (x^2 + y^2 + 1)(ax + by + c)
\]
Given \( \Delta = 0 \quad \Rightarrow \quad ax + by + c = 0 \) which represent a straight line

7.
The x-coordinate of intersection of lines \( 3x + 4y = 9 \) and \( y = mx + 1 \) is \( x = \frac{5}{3+4m} \)

For \( x \) being an integer \( 3 + 4m \) should be divisor of 5
i.e. 1, -1, 5 or -5

\[
3 + 4m = 1 \quad \Rightarrow \quad m = -\frac{1}{2} \quad \text{(Not integer)}
\]
\[
4m + 3 = -1 \quad \Rightarrow \quad m = -\frac{1}{2} \quad \text{(Intger)}
\]
\[
3 + 4m = 5 \quad \Rightarrow \quad m = -\frac{1}{2} \quad \text{(Not an integer)}
\]
\[
3 + 4m = -5 \quad \Rightarrow \quad m = -\frac{1}{2} \quad \text{(integer)}
\]
\[
\therefore \quad \text{there are two integral value of } m
8. In parallelogram OABC
   B(0,1) and point A in the point of intersection of y = mx and y = nx + 1

   \[ x = \frac{1}{m-n} \text{ and } y = \frac{m}{m-n} \]

   Now area of parallelogram = 2 (\(\triangle OAB\))

   \[ = \left| \frac{1}{2} \times \frac{1}{m-n} \right| \]

   \[ = \frac{1}{|m-n|} \]

9. \(y = |x| - 1\)
   \(y = -|x| + 1\)

   Region is clearly square with vertices at the points (1,0), (0,1), (-1,0), (0,-1). So,

   its area = \(\sqrt{2} \times \sqrt{2} = 2\).

10. Let \(\angle XOS = \alpha\) and \(\angle XOT = \frac{\alpha}{2}\)

    let \(p(\cos \theta, \sin \theta)\), then \(\angle TOP = \theta - \frac{\alpha}{2}\)

    let Q be the image of P in OT. Then \(\angle QOT = \theta - \frac{\alpha}{2}\)

    \[ \therefore \angle QOX = \theta - \alpha \]

    \[ \therefore \angle XOQ = \alpha - \theta \]

    \[ \therefore Q \text{ is image of } P \text{ in the line whose slope is } \tan \frac{\alpha}{2} \]

11. The line segment QR makes an angle 60\(^\circ\) with the positive direction of x-axis.

    hence bisector of angle POR will make 120\(^\circ\) with +ve direction of x-axis.

    \[ \therefore \text{ Its equation } \]

    \[ y = 0 = \tan 120^\circ (x - 0) \]

    \[ y = -\sqrt{3}x \]

    \[ x\sqrt{3} + y = 0 \]
12.\[ \Delta OPA \sim \Delta OQC \]
\[
\begin{align*}
\text{as } \frac{OP}{OA} &= \frac{9/4}{3} = \frac{3}{4} \\
\text{and } \frac{OQ}{OC} &= \frac{9/4}{3} = \frac{3}{4}
\end{align*}
\]

13. The line \( y = mx \) meets the given lines in \( P \left( \frac{1}{m+1}, -\frac{m}{m+1} \right) \) and \( Q \left( \frac{3}{m+1}, -\frac{3m}{m+1} \right) \). Hence equation of \( L_1 \) is
\[
y - \frac{m}{m+1} = 2 \left( x - \frac{1}{m+1} \right) \quad \Rightarrow \quad y - 2x - 1 = -\frac{3}{m+1} \quad \ldots \ldots (i)
\]
and that of \( L_2 \) is \( y - \frac{3m}{m+1} = -3 \left( x - \frac{3}{m+1} \right) \quad \Rightarrow \quad y + 3x - 3 = \frac{6}{m+1} \quad \ldots \ldots (ii)\]
Form (i) and (ii) \[ \frac{y - 2x - 1}{y + 3x - 3} = -\frac{1}{2} \quad \Rightarrow \quad x - 3y + 5 = 0; \text{ which is a straight line} \]

14. equation of line \( y - 2 = m(x - 8) \) where \( m < 0 \)
\[
\Rightarrow \quad P = \left( 8 - \frac{2}{m}, 0 \right) \quad \text{and } Q = (0, 2 - 8m)
\]
Now \[
\begin{align*}
\text{OP} + \text{OQ} &= \left| 8 - \frac{2}{m} \right| + |2 - 8m| \\
&= 10 + \frac{2}{-m} + 8(-m) \\
&\geq 10 + 2 \sqrt{\frac{2}{-m} \times 8(-m)} \\
&\geq 18
\end{align*}
\]

15. The number of integral points that lie in the interior of square OABC is 20 x 20. These points are \( (x, y) \) where \( x, y = 1, 2, \ldots, 20 \). Out of these 400 points 20 lie on the line AC. Out of the remaining exactly half lie in \( \Delta ABC \).
\[
\text{number of integral point in the triangle OAC} = \frac{1}{2} [20 \times 20 - 20] = 190
\]

**Alternative Solution**
There are 19 points that lie in the interior of \( \Delta ABC \) and on the line \( x = 1, 18 \) point that lie on the line \( x = 2 \) and so on. Thus, the number of desired points is
\[
19 + 18 + 17 + \ldots + 2 + 1 = \frac{20 \times 19}{2} = 190.
\]
16. Refer Figure
Equation of altitude BD is \( x = 3 \).

\[
\text{slope of AB} = \frac{4 - 0}{3 - 4} = -4
\]

\[ \therefore \text{slope of OE is } \frac{1}{4} \]

Equation of OE is \( y = \frac{1}{4}x \).

Lines BD and OE meets at (3, 3/4)

17. The lines given by \( x^2 - 8x + 12 = 0 \) are \( x = 2 \) and \( x = 6 \).
The lines given by \( y^2 - 14y + 45 = 0 \) are \( y = 5 \) and \( y = 9 \).
Centre of the required circle is the centre of the square.
\[ \therefore \text{Required centre is } \left( \frac{2 + 6}{2}, \frac{5 + 9}{2} \right) = (4, 7) \]

18. \( x^2 - y^2 + 2y = 1 \)
\[ x = \pm (y - 1) \]

Bisector of above lines are \( x = 0, y = 1 \)
so Area between \( x = 0, y = 1 \) and \( x + y = 3 \)
\[ = \frac{1}{2} \times 2 \times 2 = 2 \text{ squ. units} \]

19. A line passing through \( P(h, k) \) and parallel to x-axis is \( y = k \) the other lines given are \( y = x \) and \( y + x = 2 \)
Let \( ABC \) be the \( \Delta \) formed by the points of intersection of the lines (i) , (ii) and (iii) be \( A(k, k) \), \( B(1, 1) \), \( C(2 - k, k) \)
\[ \therefore \text{Area of } \Delta ABC = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 2 - k & k & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4h^2 \]
\[ C_1 \rightarrow C_1 - C_2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & k & 1 \\ 0 & 1 & 1 \end{vmatrix} = 4h^2 \]
\[ \Rightarrow \frac{1}{2} |(2 - 2k)(k - 1)| = 4h^2 \Rightarrow (k - 1)^2 = 4h^2 \]
\[ \Rightarrow k - 1 = 2h, \quad k - 1 = -2h \Rightarrow k = 2h + 1, \quad k = -2h + 1 \]
\[ \therefore \text{locus of } (h, k) \text{ is } y = 2x + 1, \quad y = -2x + 1 \]

20. \[ \begin{array}{c}
P(3,4) \\
(0,0) \\
R \end{array} \]

R is centroid hence \( R = \left( \frac{3}{3}, \frac{4}{3} \right) \)
21. \[ \frac{PR}{OQ} = \frac{OP}{QQ} \]
\[ \frac{RQ}{OQ} = \frac{2\sqrt{2}}{\sqrt{5}} \]

but statement \(-2\) is false
\[ \therefore \text{Ans. (C)} \]

22. \[ P = (-\sin (\beta - \alpha), -\cos \beta) \]
\[ Q = (\cos (\beta - \alpha), \sin \beta) \]
\[ R = (\cos (\beta - \alpha + \theta), \sin (\beta - \theta)) \quad 0 < \alpha, \beta, \theta < \frac{\pi}{4} \]

\[ x_R = \cos (\beta - \alpha) \cos \theta - \sin (\beta - \alpha) \sin \theta \]
\[ y_R = \sin \beta \cos \theta - \cos \beta \sin \theta \]

For \( P, Q, R \) to be collinear
\[ \sin \theta + \cos \theta = 1 \]
\[ \Rightarrow \sin \left( \theta + \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} \Rightarrow \text{not possible for the given interval} \theta \in \left( 0, \frac{\pi}{4} \right) \]
\[ \therefore \text{non collinear} \]

23. \[ (1 + p) x - py + p (1 + p) = 0 \quad \ldots (1) \]
\[ (1 + q) x - qy + q(1 + q) = 0 \quad \ldots (2) \]
on solving (1) and (2), we get \( C(pq, (1 + p) (1 + q)) \)
\[ \therefore \text{equation of altitude CM passing through} \ C \text{ and perpendicular to} \ AB \text{ is} \ x = pq \quad \ldots (3) \]
\[ \therefore \text{slope of line (2) is} \ \frac{1}{1+q} \]
\[ \therefore \text{slope of altitude BN (as shown in figure) is} \ \frac{-q}{1+q} \]
\[ \therefore \text{equation of BN is} \ y = -\frac{q}{1+q} (x + p) \]
\[ \Rightarrow \ y = -\frac{q}{1+q} (x + p) \quad \ldots (4) \]

Let orthocentre of triangle be \( H(h, k) \) which is the point of intersection of (3) and (4)
\[ \therefore \text{on solving (3) and (4), we get} \]
\[ x = pq \text{ and } y = -pq \quad \Rightarrow \ h = pq \text{ and } k = -pq \]
\[ \therefore \ h + k = 0 \]
\[ \therefore \text{locus of} \ (h, k) \text{ is} \ x + y = 0 \]

24. Let slope of line \( L = m \)
\[ \therefore \ \frac{m - (-\sqrt{3})}{1 + m(-\sqrt{3})} = \tan 60^\circ = \sqrt{3} \Rightarrow \frac{m + \sqrt{3}}{1 - \sqrt{3}m} = \sqrt{3} \]
taking positive sign, \( m + \sqrt{3} = \sqrt{3} - 3m \)
\[ m = 0 \]
taking negative sign
\[ m + \sqrt{3} + \sqrt{3} - 3m = 0 \]
\[ m = \sqrt{3} \]
As L cuts x-axis \( \Rightarrow m = \sqrt{3} \)
so L is \( y + 2 = \sqrt{3} (x - 3) \)

**PART - II**

1. \((h - a_1)^2 + (k - b_1)^2 = (h - a_2)^2 + (k - b_2)^2\)
\[2h(a_1 - a_2) + 2k(b_1 - b_2) + \left(\frac{a_2^2 + b_2^2 - a_1^2 - b_1^2}{2}\right) = 0\]
compare with \((a_1 - a_2)x + (b_1 - b_2)y + c = 0\)
\[c = \frac{a_2^2 + b_2^2 - a_1^2 - b_1^2}{2}\]

2. \(3h - 1 = a \cos t + b \sin t\)
\(3k = a \sin t - b \cos t\)
squaring and add. (Locus)
\((3x - 1)^2 + 9y^2 = a^2 + b^2\)

3. \(x^2 - 2pxy - y^2 = 0\)
pair of angle bisector of this pair
\[\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p}\]
\[\Rightarrow x^2 - y^2 + \frac{2}{p} xy = 0\]
compare this bisector pair with \(x^2 - 2qxy - y^2 = 0\)
\[\frac{2}{p} = -2q \Rightarrow pq = -1\]

4. **Equation of AC**
\[y - a \sin \alpha = \frac{\sin \alpha - \cos \alpha}{\cos \alpha + \sin \alpha} (x - a \cos \alpha)\]
\[y(\cos \alpha + \sin \alpha) + x(\cos \alpha - \sin \alpha) = a(\sin \alpha \cos \alpha + \sin^2 \alpha - \sin \alpha \cos \alpha + \cos^2 \alpha)\]
\[y(\cos \alpha + \sin \alpha) + x(\cos \alpha - \sin \alpha) = a.\]

5. \[G(h, k) = \left(\frac{h}{3}, \frac{k - 2}{3}\right)\]
\[\Rightarrow \frac{2h}{3} + (k - 2) = 1 \Rightarrow 2h + 3k = 9\]
Locus \(2x + 3y = 9\).

6. **Let equation of line is** \(\frac{x}{a} + \frac{y}{b} = 1\)
it passes through \((4, 3)\)
\[\frac{4}{a} + \frac{3}{b} = 1\]

---

Solutions (XI) # 57
sum of intercepts is \(-1\) \(\Rightarrow a + b = -1 \Rightarrow a = -1 - b\)

\[
\Rightarrow \frac{4}{-1-b} + \frac{3}{b} = 1
\]

\[
\Rightarrow 4b - 3 - 3b = -b - b^2
\]

\[
\Rightarrow b^2 + 2b - 3 = 0
\]

\[
\Rightarrow b = -3, 1
\]

\[
b = 1, \ a = -2
\]

\[
\frac{x}{-2} + \frac{y}{1} = 1
\]

\[
b = -3, \ a = 2
\]

\[
\frac{x}{2} + \frac{y}{-3} = 1.
\]

7. \(x^2 - 2cxy - 7y^2 = 0\)

sum of the slopes \(m_1 + m_2 = \frac{2c}{-7}\)

Product of slopes \(m_1m_2 = \frac{-1}{7}\)

given \(m_1 + m_2 = 4m_1m_2\)

\[
\Rightarrow \frac{2c}{-7} = \frac{-4}{7}
\]

\[
\Rightarrow c = 2.
\]

8. Pair \(6x^2 - xy + 4cy^2 = 0\) has its one line \(3x + 4y = 0\)

\[
\Rightarrow y = \frac{-3x}{4} + \frac{3x^2}{4} + \frac{4c}{9} + \frac{x^2}{16} = 0
\]

\[
\Rightarrow 24x^2 + 3x^2 + 9cx^2 = 0
\]

\[
\Rightarrow c = -3.
\]

9. \(ax + 2by + 3b = 0\)

\(bx - 2ay - 3a = 0\)

\[
\frac{x}{-6ab + 6ab} = \frac{y}{3b^2 + 3a^2} = \frac{1}{-2a^2 - 2b^2}
\]

Hence point of intersection \((0, -3/2)\)

Line parallel to x-axis \(y = -3/2\).

10. \(\therefore a, b, c\) are in H.P.

\[
\frac{2}{b} = \frac{1}{a} + \frac{1}{c}
\]

\[
\Rightarrow \frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0
\]

given line \(\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0\)

Clearly line passes through \((1, -2)\).

11. Centroid is \(\left(1, \frac{7}{3}\right)\)
12. Pair of lines \( ax^2 + 2(a + b)xy + by^2 = 0 \)

Area of sector \( A_1 = \frac{1}{2} r^2 \theta_1 \)

\[ A_2 = \frac{1}{2} r^2 \theta_2 \]

\( \theta_1 + \theta_2 = 180^\circ \)

given \( A_1 = 3A_2 \Rightarrow \theta_2 = 30^\circ \)

\[ \Rightarrow \theta_2 = 45^\circ, \theta_1 = 135^\circ \]

Angle between lines is \( = \frac{2\sqrt{(a + b)^2 - ab}}{a + b} = 1 \)

\[ \Rightarrow 4(a^2 + b^2 + ab) = a^2 + b^2 + 2ab \]

\[ \Rightarrow 3a^2 + 3b^2 + 2ab = 0. \]

13. Let equation of line is \( \frac{x}{a} + \frac{y}{b} = 1. \)

By section formula

\[ a = 3 \Rightarrow a = 6 \]

\[ b = 4 \Rightarrow b = 8 \]

\[ \frac{x}{6} + \frac{y}{8} = 1 \Rightarrow 4x + 3y = 24. \]

14. Since \((1, 1)\) and \((a, a^2)\) Both lies same side with respect to both lines \( a - 2a^2 < 0 \Rightarrow 2a^2 - a > 0 \)

\[ \Rightarrow a(2a - 1) > 0 \]

\( a \in (-\infty, 0) \cup \left( \frac{1}{2}, \infty \right) \)

\( 3a - a^2 > 0 \Rightarrow a^2 - 3a < 0 \Rightarrow a \in (0, 3) \)

Hence after taking intersection \( a \in \left( \frac{1}{2}, 3 \right) \).

15. \( AB = \sqrt{(h-1)^2 + (k-1)^2} \)

\( BC = 1 \)

\( AC = \sqrt{(h-2)^2 + (k-1)^2} \)

\( AB^2 + BC^2 = AC^2 \Rightarrow (h-1)^2 + (k-1)^2 + 1 = (h-2)^2 + (k-1)^2 \)

\[ \Rightarrow 2h = 2 \Rightarrow h = 1 \]

Area of \( \triangle ABC = \frac{1}{2} \sqrt{(h-1)^2 + (k-1)^2} \times 1 = 1 \)

\( (k-1)^2 = 4 \Rightarrow k-1 = \pm 2 \Rightarrow k = 3, -1. \)

16. \( P(-1,0) \qquad Q(0,0) \)

\( \angle R(3,3;3) \)

\( \angle 60^\circ \)
The line segment \( QR \) makes an angle 60° with the positive direction of \( x \)-axis. hence bisector of angle \( PQR \) will make 120° with +ve direction of \( x \)-axis.

\[
\therefore \text{Its equation} = y - 0 = \tan 120^\circ (x - 0) \\
y = -\sqrt{3}x \\
x\sqrt{3} + y = 0
\]

17. Bisector of \( x = 0 \) and \( y = 0 \) is either \( y = x \) or \( y = -x \)
If \( y = x \) is Bisector, then 
\[
mx^2 + (1 - m^2)x^2 - mx = 0 \\
\Rightarrow m + 1 - m^2 - m = 0 \\
\Rightarrow m^2 = 1 \\
\Rightarrow m = \pm 1.
\]

18. Slope of \( PQ \) = \( \frac{1}{1-k} \)

Hence equation of \( \perp \) to line \( PQ \)
\[
y - \frac{7}{2} = (k - 1) \left( x - \frac{(1+k)}{2} \right)
\]
Put \( x = 0 \)
\[
y = \frac{7}{2} + \frac{1-k}{2} \cdot \frac{1+k}{2} = -4 \\
7 + (1 - k^2) = -8 \Rightarrow k^2 = 16 \Rightarrow k = \pm 4.
\]
Hence possible answer = \( -4 \).

19. \( p(p^2 + 1) x - y + q = 0 \)
\( (p^2 + 1)^2 x + (p^2 + 1) y + 2q = 0 \) are perpendicular for a common line

\[
\Rightarrow \text{lines are parallel} \quad \Rightarrow \text{slopes are equal}
\]
\[
\therefore \quad \frac{p(p^2 + 1)}{1} = -\frac{(p^2 + 1)^2}{p^2 + 1} \quad \Rightarrow \quad p = -1
\]

20. \( \therefore \quad \frac{PA'}{PB'} = \frac{3}{1} \\
\therefore \quad (x + 1)^2 + y^2 = 9((x - 1)^2 + y^2) \\
x^2 + 2x + 1 + y^2 = 9x^2 + 9y^2 - 18x + 9 \\
8x^2 + 8y^2 - 20x + 8 = 0 \\
x^2 + y^2 = -\frac{10}{4}x + 1 = 0 \\
\therefore \quad \text{circumcentre} \left( \frac{5}{4}, 0 \right) .
\]

21. \( \frac{x}{5} + \frac{y}{b} = 1 \)
\[
\frac{13}{5} + \frac{32}{b} = 1 \quad \Rightarrow \quad \frac{32}{b} = -\frac{8}{5} \quad \Rightarrow \quad b = -20
\]
\[
\frac{x}{5} - \frac{y}{20} = 1 \quad \Rightarrow \quad 4x - y = 20
\]
Line \( K \) has same slope \( \Rightarrow \quad \frac{3}{c} = 4 \\
c = -\frac{3}{4} \quad \Rightarrow \quad 4x - y = -3
\]
\[
\text{distance} = \frac{23}{\sqrt{17}}
\]

Hence correct option is \( 3 \)
22. AD : DB = 2√2 : √5
   ∴ OD is angle bisector of angle AOB
   ∴ St. 1 true
   St. 2 false (obvious) Ans.

23. x + y = |a|
    ax - y = 1
    if a > 0
    x + y = a
    ax - y = 1

    x(1 + a) = 1 + a  as x = 1
    y = a - 1
    It is in the first quadrant
    so a - 1 ≥ 0
    a ≥ 1
    a ∈ [1, ∞)
    If a < 0
    x + y = -a
    ax - y = 1

    x(1 + a) = 1 - a
    x = \frac{1-a}{1+a} > 0  \Rightarrow \frac{a-1}{a+1} < 0
    y = -a - \frac{1-a}{1+a}
    = -a - \frac{a^2 - 1 + a}{1+a} > 0
    - \left( \frac{a^2 + 1}{a+1} \right) > 0  \Rightarrow \frac{a^2 + 1}{a+1} < 0
    from (1) and (2)  a ∈ {∅}

24. α = 3h
    β - 2 = 3k
    β = 3k + 2
    third vertex on the line 2x + 3y = 9
    2α + 3β = 9
    2(3h) + 3(3k + 2) = 9
    2h + 3k = 1
    2x + 3y - 1 = 0
25. \[ A(1, 1) \quad B(2, 4) \]
\[ C \left( \frac{8}{5}, \frac{14}{5} \right) \]
\[ \therefore C \left( \frac{8}{5}, \frac{14}{5} \right) \]
Line \( 2x + y = k \) passes \( C \left( \frac{8}{5}, \frac{14}{5} \right) \)
\[ \frac{2 \times 8}{5} + \frac{14}{5} = k \]
\[ k = 6 \]

26. \( (y - 2) = m(x - 1) \)
\[ OP = 1 - \frac{2}{m} \]
\[ OQ = 2 - m \]
Area of \( \triangle POQ \) = \( \frac{1}{2} (OP)(OQ) = \frac{1}{2} \left( 1 - \frac{2}{m} \right) (2 - m) \]
\[ = \frac{1}{2} \left[ 2 - m - \frac{4}{m} + 2 \right] \]
\[ = \frac{1}{2} \left[ 4 - \left( m + \frac{4}{m} \right) \right] \]
\[ m = -2 \]

**ADVANCE LEVEL PROBLEMS**

**PART - I**

1. Condition for concurrency
\[
\begin{vmatrix}
1 & 2a & a \\
1 & 3b & b \\
1 & 4c & c
\end{vmatrix} = 0 \quad \Rightarrow \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c}
\]
So \( a, b, c \) are in H.P.

2. \( x^2(\sec^2 \theta - \sin^2 \theta) - 2xy \tan \theta + y^2 \sin^2 \theta = 0 \)
\[ |m_1 - m_2| = \sqrt{(m_1 + m_2)^2 - 4m_1m_2} \]
\[ \sqrt{\left( \frac{2 \tan \theta}{\sin^2 \theta} \right)^2 - 4 \left( \frac{\sec^2 \theta - \sin^2 \theta}{\sin^2 \theta} \right)} = 2 \]

3. Equation of family of curves passing through intersection of \( C_1 \) & \( C_2 \) is
\[ \lambda x^2 + 4y^2 = 2xy - 9x + 3 + \mu(2x^2 + 3y^2 - 4xy + 3x - 1) = 0 \]  
\[ \text{.................(i)} \]
It will give the joint equation of pair of lines passing through origin, if coefficient of \( x = 0 \) & Constant = 0
\[ \Rightarrow \mu = 3 \]
put \( \mu = 3 \) in equation (i), we get
\[ \lambda x^2 + 4y^2 = 2xy + 6x^2 + 9y^2 - 12xy = 0 \]
It will subtend 90° at origin if coeff. of \( x^2 \) + coeff. of \( y^2 \) = 0
\[ \Rightarrow \lambda = -19 \]
4. For B and C apply Parametric form
\[ \frac{x - 3}{\cos \theta} = \frac{y - 2}{\sin \theta} = \pm 5 \]
Points are (7, 5) & (−1, −1)

5. From figure it is clear that A is orthocentre of ΔABC

6. \( px^2 - qxy - y^2 = 0 \)
\( m_1 = \tan \alpha, \quad m_2 = \tan \beta \)
\( m_1 + m_2 = -q, \quad m_1 m_2 = -p \)
\[ \Rightarrow \tan (\alpha + \beta) = -\frac{q}{1 + p} \]

7. \( (2 + \lambda) x + (1 - 2\lambda) y + (4 - 3\lambda) = 0 \)
Distance from point A is
\[ \left[ \frac{(2 + \lambda) \times 2 - 3(1 - 2\lambda) + (4 - 3\lambda)}{\sqrt{(2 + \lambda)^2 + (1 - 2\lambda)^2}} \right] = \sqrt{10} \]
\[ \Rightarrow \lambda = 1 \]
Hence, the required line is \( 3x - y + 1 = 0 \)

8. To find equations of AB and CD
\[ \therefore AB \text{ and } CD \text{ are parallel to } 3x - 4y = 0 \text{ and at a distance of 2 units from } (1, 1) \]
\[ \therefore 3x - 4y + k = 0 \]
and
\[ \frac{3 - 4 + k}{5} = 2 \quad \Rightarrow \quad k - 1 = \pm 10 \]
\[ \Rightarrow \quad k = 11, -9 \]
\[ \therefore \text{equations of two sides of the square which are parallel to } 3x - 4y = 0 \text{ are} \]
\[ 3x - 4y + 11 = 0 \text{ and } 3x - 4y - 9 = 0 \]
Now the remaining two sides will be perpendicular to
\[ 3x - 4y = 0 \text{ and at a distance of 2 unit from } (1, 1) \quad \therefore \]
\[ 4x + 3y + k = 0 \]
and
\[ \frac{4 + 3 + k}{5} = 2 \quad \Rightarrow \quad k + 7 = \pm 10 \]
\[ \Rightarrow \quad k = 3, -17 \quad \therefore \text{remaining two sides are} \]
\[ 4x + 3y + 3 = 0 \quad \text{and} \quad 4x + 3y - 17 = 0 \]

9. Given
\[ x \cos \alpha + y \sin \alpha = a \quad \text{......(1)} \]
\[ x \sin \alpha - y \cos \alpha = b \quad \text{......(2)} \]
Square (1) and (2) then add them.
\[ x^2 + y^2 = a^2 + b^2 \]
10. Let point of concurrency of given family of lines is Q and it can be obtained by solving
   \(3x + 4y + 6 = 0\)
   and \(x + y + 2 = 0\) :: \(Q = (-2, 0)\)

   Now required line will pass through Q\((-2, 0)\) and perpendicular to PQ.

   :: Equation of required line is
   \[y - 0 = \frac{-4}{3}(x + 2) \quad \Rightarrow \quad 4x + 3y + 8 = 0\]

11. (i) After reflection about line \(y = x\) position of point will be \((1, 4)\)
    (ii) After this step \((3, 4)\)
    (iii) \((h + ki) = (3 + 4i) e^{i \pi/4} = (3 + 4i) \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \Rightarrow h = -\frac{1}{\sqrt{2}}, k = \frac{7}{\sqrt{2}}\)

   Hence the final position will be \((-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}})\)

12. Let the point P be 
   \(|x| + |y| = 3 \quad \text{.....(i)}\)

   **Case - 1** \(x > 0, y > 0\)

   Equation (i) will become :
   \(x + y = 3\)

   Similarly for each quadrant, a triangle will be formed. Hence area enclosed will be 18.

13. \(\therefore \quad P = (-4, -2)\)
    and \(Q = (-2, -6)\)

   \(\therefore \quad \) Let slopes of PM and QM be \(m_1\) and \(m_2\) respectively.

   \(\therefore \quad m_1 = 3 \text{ and } m_2 = \frac{1}{2}\).

   Let ‘\(\theta\)’ be the acute angle between PM and QM

   \(\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}\)
14. For collinearity of 3 points
\[
\begin{vmatrix}
-2 & 0 & 1 \\
-1 & 1 & 1 \\
\cos \theta & \sqrt{3} \sin \theta & 1
\end{vmatrix} = 0
\]
\[\Rightarrow \sqrt{3} \sin \theta - \cos \theta = 2 \quad \Rightarrow \quad \theta = \frac{2\pi}{3}\]

15. For \(\triangle ABC\)
\[a + b > c, \quad b + c > a, \quad c + a > b\]
\[x^2 + 4x + 3 > x^2 + 3x + 8 \quad \Rightarrow \quad x > 5\]
\[x^2 + 5x + 11 > x^2 + 2x \quad \Rightarrow \quad x > \frac{-11}{3}\]
\[2x^2 + 5x + 8 > 2x + 3 \quad \Rightarrow \quad x \in R\]
Common to all is \(x > 5\).

16. \(\therefore\) point of intersection of the two ray is \(P(0, 2)\)
\[\therefore\] Point A is \(\left(\frac{2}{\sqrt{3}}, 0\right)\) or \(\left(-\frac{2}{\sqrt{3}}, 0\right)\)
and PO is bisector of the angle between two rays
\(\therefore\) required point is \((0, 0)\)

17. Slope BD = -1.
Equation of BD, \(x + y = a + b\)
equation of AC = \(x - y = a\)
On solving, we get \(O = \left(a + \frac{b}{2}, \frac{b}{2}\right)\)
\(B = (a + b, 0)\)
m\(\text{AC} = 1 = \tan \theta\)
\(OA = OD = \frac{b}{\sqrt{2}}\)
Apply parametric form for finding A & C
\[x - \left(a + \frac{b}{2}\right) = \frac{1}{\sqrt{2}} \quad \frac{y - \frac{b}{2}}{1} = \pm \frac{b}{\sqrt{2}}\]
A & C are \((a, 0)\) and \((a + b, b)\)

18. \[
\begin{vmatrix}
p & q & r \\
q & r & p \\
r & p & q
\end{vmatrix} = 0
\]
\[p^3 + q^3 + r^3 - 3pqr = 0 \quad \Rightarrow \quad (p + q + r) (p^2 + q^2 + r^2 - pq - qr - rp) = 0\]
19. \( k_1u - k_2v = 0 \quad \ldots \quad (i) \)
\( k_1u + k_2v = 0 \quad \ldots \quad (ii) \)

\[ \begin{align*}
\Rightarrow \quad k_1u - k_2v &= \pm (k_1u + k_2v) \\
\sqrt{(ak_1 - bk_2)^2 + (k_1b + ak_2)^2} &= \sqrt{(k_1a + bk_2)^2 + (k_1b - ak_2)^2} \\
(\text{by taking positive sign in (iii), we get}) \\
k_1u - k_2v &= k_1u + k_2v \quad \Rightarrow \quad v = 0 \\
(\text{by taking negative sign in (iii), we get}) \\
k_1u - k_2v &= 2k_2v \quad \Rightarrow \quad u = 0
\end{align*} \]

**PART - II**

1. \( \frac{x_1 - 1}{\cos \theta} = \frac{y_1 - 2}{\sin \theta} = \pm \frac{\sqrt{6}}{3} \)
\((\because (1, 2) \text{ lie below the line})\)

\[ x_1 = 1 + \frac{\sqrt{6}}{3} \cos \theta, \quad y_1 = 2 + \frac{\sqrt{6}}{3} \sin \theta \]
\((x_1, y_1) \text{ lies on } x + y = 4\)

\[ 3 + \frac{\sqrt{6}}{3} (\sin \theta + \cos \theta) = 4 \]

\[ \sin \theta + \cos \theta = \frac{3}{\sqrt{6}} = \frac{3}{\sqrt{3} \sqrt{2}} \]

\[ \frac{1}{\sqrt{2}} (\sin \theta + \cos \theta) = \frac{3}{2 \sqrt{2}} \cdot \frac{1}{\sqrt{2}} \]

\[ \sin (\theta + \frac{\pi}{4}) = \frac{\sqrt{3}}{2} = \sin 60^\circ \text{ or } \sin 120^\circ \]

\[ \theta = \frac{\pi}{12}, \quad \frac{5\pi}{12} \quad \left( -\frac{\sqrt{6}}{3} \text{ is rejected} \right) \]

2. \[ \tan \theta = \frac{4 - 3}{3 - 4} = \frac{7}{24} \]
\[ \csc \theta = \frac{25}{7} \]
P₁ is ⊥ from AB to CD, P₂ is ⊥ from AD to BC
for finding P₁ choose arbitrary point (a, a) on AB
\[ P₁ = \frac{4a - 3a - 3a}{5} = \frac{2a}{5} \]
for P₂ choose arbitrary point (a/2, 0) on AD
\[ P₂ = \frac{|0 - 2a + a|}{5} = \frac{a}{5} \]
∴ Area = \( P₁P₂ \csc \theta = \frac{2a^2}{7} \)

3. Slope of BC is
\[ \frac{-3 - (-1)}{-1 - (-3)} = \frac{-2}{2} = -1 \]
∴ Equation of a line parallel to BC is
\[ y = -x + c \]
i.e. \( x + y = c \)

its distance from the origin is
\[ \frac{|c|}{\sqrt{2}} = \frac{1}{2} \]
∴ Equations of the lines are
\[ x + y \pm \frac{1}{\sqrt{2}} = 0 \]
Since the required line intersects OB and OC, therefore, it is the line whose y intercept is negative. Hence
the required line is \( x + y + \frac{1}{\sqrt{2}} = 0 \).

4. \( AB = \sqrt{64 + 361} = \sqrt{425} \)
\( BC = \sqrt{576 + 36} = \sqrt{612} \)
\( AC = \sqrt{256 + 169} = \sqrt{425} \)
\( AB = AC \neq BC \)
∴ triangle is isosceles
and in isosceles triangle O, H, I, G are collinear

5. D is mid point of AB and lies on the line \( 3x + y = 6\lambda \)
\[ \Rightarrow 3 \cdot \frac{\lambda^2 + \lambda + 1}{2} + \frac{2\lambda - 1}{2} = 6\lambda \]
\[ 3\lambda^2 - 7\lambda + 2 = 0 \]
\[ \lambda = \frac{1}{3}, 2 \]
multiplication of slope of AB & line = -1
\[ \frac{-1}{\lambda - \lambda^2 - 1} = -1 \]
\[ \lambda^2 - \lambda - 2 = 0 \]
\[ \lambda = -1, 2 \]
\[ \lambda = 2 \text{ satisfies both (1) & (2)} \]

6. \( AB = AC \)
The bisectors are \[
\frac{7x - y + 3}{5\sqrt{2}} = \pm \frac{(x + y - 3)}{\sqrt{2}}
\]
Their slopes are \(\frac{1}{3}, -3\)

Required lines are \(y + 10 = \frac{1}{3} (x - 1)\) and \(y + 10 = -3(x - 1)\)
i.e. \(x - 3y - 31 = 0\) and \(3x + y + 7 = 0\)

\[
\therefore AB = 2d
\]
\[
\therefore OAPB is a cyclic quadrilateral and OP will be diameter of the circumcircle of this quadrilateral
\]
Let Q be the centre of the circle
\[
\therefore \sin \theta = \frac{2d}{\sqrt{x_1^2 + y_1^2}} \quad \ldots \quad (i)
\]
\[
\therefore \tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b} \quad \ldots \quad (ii)
\]
\[
\therefore \text{from (i) and (ii), we get}
\]
\[
2\sqrt{h^2 - ab} = \frac{2d}{\sqrt{x_1^2 + y_1^2 - 4d^2}}
\]
\[
\therefore \text{Locus of } P(x_1, y_1) \text{ is}
\]
\[
(x^2 + y^2) ((h^2 - ab) = d^2 ((a - b)^2 + 4h^2)
\]

8. \(\because\) The slopes of the lines AB, BC and CA are \(-1, -\frac{1}{7}\) and \(-7\) respectively

Let \(m_1 = -\frac{1}{7}, m_2 = -1, m_3 = -7\)
\(\therefore\) \(m_1 > m_2 > m_3\)
\(\therefore\) tangent of internal angles of the triangle are
\[
\tan A = \frac{3}{4}, \tan B = \frac{3}{4} \text{ and } \tan C = -\frac{24}{7}
\]
\(\therefore\) interior angles A and B are acute and interior angle C is obtuse
\(\therefore\) internal bisector of B = acute bisector of B= \(3x + 6y - 16 = 0\)
External bisector of C = acute bisector of C = \(8x + 8y + 7 = 0\)
Internal bisector of A = acute bisector of A = \(12x + 6y - 11 = 0\)

9. Equation of line passing through P(\(-1, 2\)) making angle \(\theta\) with +ve direction of x-axis is given by
\[
\frac{x + 1}{\cos \theta} = \frac{y - 2}{\sin \theta} = r_1, r_2, r_3 \quad \text{(parametric form)}
\]
where \(r_1, r_2, r_3\) are distances of points A, Q, B from point P respectively.
Hence coordinates of A(r_1 \cos \theta - 1, r_1 \sin \theta + 2)
But A lies on x-axis

Hence \( r_1 \sin \theta + 2 = 0 \) \( \Rightarrow \) \( \sin \theta = -\frac{2}{r_1} \)

coordinates of point B \( (r_3 \cos \theta - 1, r_3 \sin \theta + 2) \)

Point B lies on y-axis hence

\( r_3 \cos \theta - 1 = 0 \) \( \Rightarrow \) \( \cos \theta = \frac{1}{r_3} \)

Coordinates of point Q \( (r_2 \cos \theta - 1, r_2 \sin \theta + 2) \)

Hence \( h = r_2 \cos \theta - 1 \) \( \Rightarrow \) \( \cos \theta = \frac{h + 1}{r_2} \)

and \( k = r_2 \sin \theta + 2 \) \( \Rightarrow \) \( \sin \theta = \frac{k - 2}{r_2} \)

Now given that \( r_1, r_2, r_3 \) are in H.P.

\[
\frac{2}{r_2} = \frac{1}{r_1} + \frac{1}{r_3}
\]

\[
\frac{2}{r_2} = -\frac{\sin \theta}{2} + \cos \theta
\]

\[
\Rightarrow \quad \frac{2}{r_2} = -\frac{1}{2} \left( \frac{k - 2}{r_2} \right) + \frac{(h + 1)}{r_2} \quad \Rightarrow \quad 2 = -\frac{1}{2} (k - 2) + (h + 1)
\]

\[
\Rightarrow \quad 4 = -k + 2 + 2h + 2 \quad \Rightarrow \quad 2h = k
\]

locus \( y = 2x \)

Alt. : Use P and Q are harmonic conjugates with respect to A and B.

10. Let \( P(h, k) \) be a variable point on the lines passing through the origin.

\[
\Rightarrow \quad (kx_i - hy_i)^2 = \delta^2 (h^2 + k^2)
\]

\[
\Rightarrow \quad \text{locus of } P(h, k) \text{ is } (x, y - xy) = \delta^2 (x^2 + y^2)
\]

solving it, we get

\[
(y_i^2 - \delta^2) x^2 - 2x_i y_i xy + (x_i^2 - \delta^2) y^2 = 0.
\]

11. Let the line \( L \) through the origin is

\[
x = r \cos \theta \]

\[
y = r \sin \theta
\]

as \( L \) intersects \( L_1 \) at \( Q \) and \( OQ = r_1 \)

\[
r_1 \sin \theta = m_1 r_1 \cos \theta + c_1
\]

similarly, \( L \) intersects \( L_2 \) at \( R \) and \( OR = r_2 \)

\[
r_2 \sin \theta = m_2 r_2 \cos \theta + c_2
\]

Let \( P = (h, k) \) & \( OP = r \)

\[
r_1 = r_1 \]

\[
h = r \cos \theta
\]

\[
k = r \sin \theta
\]

putting the values of \( r_1 \) and \( r_2 \) from (1) and (2) in (3)

\[
r^2 = \frac{c_1}{(\sin \theta - m_1 \cos \theta)} \cdot \frac{c_2}{(\sin \theta - m_2 \cos \theta)}
\]

putting the value of \( \cos \theta \) and \( \sin \theta \) from (4) and (5) in (6), we get

\[
r^2 = \frac{c_1 c_2}{r (k - m_1 h) (k - m_2 h)} \quad \Rightarrow \quad (k - m_1 h) (k - m_2 h) = c_1 c_2
\]

replacing \( (h, k) \) by \( (x, y) \) we get the desired locus

\[
(y - m_1 x) (y - m_2 x) = c_1 c_2
\]
12. take any point on line
\[ 3x + 2y + 4 = 0 \]
put \( x = 0 \), we get \( y = -2 \)
Now image of \((0, -2)\) in line \(2x + 3y + 1 = 0\)
\[
\frac{x - 0}{2} = \frac{y + 2}{3} = -2 \left( \frac{0 - 6 + 1}{4 + 9} \right) = \frac{10}{13}
\]
Hence \( x = \frac{20}{13} \) and \( y = \frac{30}{13} - 2 = \frac{4}{13} \)
Point of intersection of \(2x + 3y + 1 = 0\) and \(3x + 2y + 4 = 0\) is \((-2, 1)\)
Hence equation of other line
\[
y - \frac{4}{13} = \frac{4/13 - 1}{20/13 + 2} \left( x - \frac{20}{13} \right)
\]
After simplification, we get \(9x + 46y = 28\)

---

**CIRCLE**

**EXERCISE # 1**

**PART - I**

**Section (A) :**

A-2. Since BD is diameter of circle
Hence \((x - a)(x - 0) + (y - 0)(y - a) = 0\)
\[ \Rightarrow x^2 + y^2 = a(x + y) \]

A-6. \( x = -3 + 2\sin \theta \) \( \Rightarrow \) \( x + 3 = 2 \sin \theta \)
\( y = 4 + 2\cos \theta \) \( \Rightarrow \) \( y - 4 = 2 \cos \theta \)
Squarring and add \((x + 3)^2 + (y - 4)^2 = 4\)

**Section (B) :**

B-4. \( S_1 = (9)^2 + (0)^2 - 16 = 65 > 0 \)
Since \((9, 0)\) lies outside the circle. Hence two real tangents can be drawn.
Now \( S = x^2 + y^2 - 16 \)
\( S_1 = 9x - 16 \)
Hence pair of tangents \( SS_1 = T^2 \)
\((x^2 + y^2 - 16)(65) = (9x - 16)^2 \)
\( 65x^2 + 65y^2 - 1040 = 81x^2 + 256 - 288x \)
\( 16x^2 - 65y^2 - 288x + 1296 = 0 \)

Angle between these tangents
\[
\angle \text{between these tangents} = \frac{2\sqrt{h^2 - ab}}{(a + b)} = \frac{2\sqrt{0 + 16 \times 65}}{16 - 65} = \frac{8\sqrt{65}}{49}
\]

B-5. given \( \sqrt{f^2 + g^2 - 6} = 2\sqrt{f^2 + g^2 + 3f + 3g} \)
\[ \Rightarrow 3g^2 + 3f^2 + 12g + 12f + 6 = 0 \]
\[ \Rightarrow g^2 + f^2 + 4g + 4f + 2 = 0 \]

**Section (C) :**

C-4. Area of triangle formed by pair of tangents & chord of contact is \( \frac{RL^2}{R^2 + L^2} \)
Here \( R = a \)
\[ L = \sqrt{h^2 + k^2 - a^2} \]
Hence Area \[ \frac{a\left(h^2 + k^2 - a^2\right)^{3/2}}{(h^2 + k^2)^{3/2}} \]
\[ C-7. \quad T = S_1 \\
-2x - 3y + 3(x - 2) + 4(y - 3) + 9 \\
= 4 + 9 - 12 - 24 + 9 \\
x + y + 5 = 0 \]

Section (D) :

D-1. \[ S_1 : x^2 + y^2 - 2x - 6y + 9 = 0 \quad C_1(1, 3), r_1 = 1 \]
\[ S_2 : x^2 + y^2 + 6x - 2y + 1 = 0 \quad C_2(-3, 1), r_2 = 3 \]
\[ C_1C_2 = \sqrt{16 + 4} = \sqrt{20} \]
\[ n + r_2 = 4 \]
Hence \( C_1C_2 > r_1 + r_2 \) Both circles are non-intersecting.
Hence there are four common tangents.

**Transverse common tangents :**

coordinate of \( P \)
\[ \left( \frac{3 - 3}{1 + 3}, \frac{1 + 9}{1 + 3} \right) = \left( 0, \frac{5}{2} \right) \]
Let slope of these tangents is \( m \)
\[ y - \frac{5}{2} = mx \Rightarrow mx - y + \frac{5}{2} = 0 \]
Now \[ \left| \frac{m - 3 + \frac{5}{2}}{\sqrt{1 + m^2}} \right| = 1 \Rightarrow \left| \frac{m - \frac{1}{2}}{\sqrt{1 + m^2}} \right| = \frac{1}{\sqrt{1 + m^2}} \]
\[ \Rightarrow m^2 + 1 - m = 1 + m^2 \Rightarrow m = -\frac{3}{4} \]
other tangents is vertical
Equation of tangents \( x = 0 \)
\[ -\frac{3}{4}x - y + \frac{5}{2} = 0 \Rightarrow -3x - 4y + 10 = 0 \Rightarrow 3x + 3y = 10 \]

**Direct common tangents**

coordinate of \( Q \)
\[ \left( \frac{-3 - 3}{1 - 3}, \frac{1 - 9}{1 - 3} \right) = Q(3, 4) \]
Hence equations \( y - 4 = m(x - 3) \Rightarrow mx - y + (4 - 3m) = 0 \)
\[ \Rightarrow \left| \frac{m - 3 + 4 - 3m}{\sqrt{1 + m^2}} \right| = 1 \]
\[ \Rightarrow |1 - 2m| = \sqrt{1 + m^2} \]
\[ \Rightarrow 1 + 4m^2 - 4m = 1 + m^2 \Rightarrow 3m^2 - 4m = 0 \Rightarrow m = 0, \frac{4}{3} \]
Hence equation \( y - 4 = 0(x - 3) \Rightarrow y = 4 \)
\[ y - 4 = \frac{4}{3}(x - 3) \Rightarrow 4x - 3y = 0 \]

D-3. \quad Equation of circle passing through origin is \( x^2 + y^2 + 2gx + 2fy = 0 \)
This circle cuts the circle \( x^2 + y^2 - 4x + 6y + 10 = 0 \) orthogonally
\[ 2g(-2) + 2f(3) = 0 + 10 \]
\[ \Rightarrow -2g + 3f - 5 = 0 \]
& \( x^2 + y^2 + 12y + 6 = 0 \) also
\[ 2g(0) + 2f(6) = 6 + 0 \Rightarrow f = \frac{1}{2} \]
\[-2g + \frac{3}{2} - 5 = 0 \Rightarrow 2g = -\frac{7}{2} \Rightarrow g = -\frac{7}{4}\]

Hence circle \( x^2 + y^2 + 2 \left( -\frac{7}{4} \right) x + 2 \left( \frac{1}{2} \right) y = 0 \)

\[2x^2 + 2y^2 - 7x + 2y = 0\]

**Section (E):**

**E-1.** Equation of circumcircle of this triangle

\[
L_1 L_2 + \lambda L_2 L_3 + \mu L_3 L_1 = 0
\]

\[
(x + 2y - 5)(x + y - 6) + \lambda (x + y - 6)(2x + y - 4) + \mu (x + 2y - 5)(2x + y - 4) = 0
\]

coef. of \( xy \) = 0 \( \Rightarrow 3 + 3\lambda + 5\mu = 0 \Rightarrow 3\lambda + 5\mu = 3 = 0 \) \( \ldots (1) \)

coef. \( x^2 \) = coef. \( y^2 \) \( \Rightarrow 1 + 2\lambda + 2\mu = 2 + \lambda + 2\mu \)

\[\Rightarrow \lambda = 1 \quad \mu = -\frac{6}{5}\]

Hence \( (x + 2y - 5)(x + y - 6) + (x + y - 6)(2x + y - 4) - \frac{6}{5}(x + 2y - 5)(2x + y - 4) = 0 \)

\[\Rightarrow x^2 + y^2 - 17x - 19y + 50 = 0\]

**E-2.**

\[x^2 + y^2 - 10x + \lambda (2x - y) = 0 \quad \ldots (i)\]

\[x^2 + y^2 + 2x (\lambda - 5) - \lambda y = 0\]

Centre \((- (\lambda - 5), \lambda/2)\)

Using on \( y = 2x\)

\[\frac{\lambda}{2} = -2(\lambda - 5)\]

\[\frac{5\lambda}{2} = 10\]

Putting \( \lambda = 4\)

\[x^2 + y^2 - 2x - 4y = 0\]

**PART - II**

**Section (A):**

**A-1.**

Diameter = \( 4\sqrt{2} \)

\( r = 2\sqrt{2} \)

**A-6.**

Let equation of required circle is

\[x^2 + y^2 + 2gx + 2fy + c = 0\]

It passes through \((1, -2)\) & \((3, -4)\)

\[2g - 4f + c = -5\]

\[6g - 8f + c = -25\]

\[4g - 8f + 2c = -10\]

\[6g - 8f + c = -25\]

\[-2g + c = 15\]

Circle touches \( x \)-axis \( g^2 = c \) \( \Rightarrow g^2 - 2g - 15 = 0 \)

\( g = 5, -3 \)

\( g = 5, \quad c = 25, \quad f = 10 \) \( \Rightarrow x^2 + y^2 + 10x + 20y + 25 = 0 \)

\( g = -3, \quad c = 9, \quad f = 2 \) \( \Rightarrow x^2 + y^2 - 6x + 4y + 9 = 0 \)
Section (B) :
B-1. Point on the line \(x + y + 13 = 0\) nearest to the circle \(x^2 + y^2 + 4x + 6y - 5 = 0\) is foot of \(\perp\) from centre
\[
\begin{align*}
\frac{x + 2}{1} &= \frac{y + 3}{1} = -\left(\frac{-2 - 3 + 13}{1^2 + 1^2}\right) = -4 \\
x &= -6 \quad y = -7
\end{align*}
\]
B-3. Let slope of required line is \(m\)
\[
y - 3 = m(x - 2) \\
\Rightarrow mx - y + (3 - 2m) = 0
\]
length of \(\perp\) from origin
\[
= 3
\]
\[
\Rightarrow 9 + 4m^2 - 12m = 9 + 9m^2
\]
\[
\Rightarrow 5m^2 + 12m = 0
\]
\[
\Rightarrow m = 0, \quad -\frac{12}{5}
\]
Hence lines are \(y - 3 = 0\) \(\Rightarrow y = 3\)
\(y - 3 = -\frac{12}{5}(x - 2) \Rightarrow 5y - 15 = -12x + 24 \quad \Rightarrow 12x + 5y = 39.\)
B-5. Line parallel to given line \(4x + 3y + 5 = 0\) is \(4x + 3y + k = 0\) This is tangent to \(x^2 + y^2 - 6x + 4y - 12 = 0\)
\[
\left|\frac{12 - 6 + k}{5}\right| = 5
\]
\[
6 + k = \pm 25 \Rightarrow k = 19, -31
\]
Hence required line \(4x + 3y - 31 = 0, \quad 4x + 3y + 19 = 0\)
B-9. As we know
\[
PA.PB = PT^2 = (\text{Length of tangent})^2
\]
Length of tangent = \(\sqrt{16 \times 9} = 12\)
B-10. Let any point on the circle \(x^2 + y^2 + 2gx + 2fy + p = 0\) \((\alpha, \beta)\)
This point satisfies \(\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + p = 0\)
Length of tangent from this point to circle \(x^2 + y^2 + 2gx + 2fy + q = 0\)
\[
\text{length} = \sqrt{S_1} = \sqrt{\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + q} = \sqrt{q - p}
\]
Section (C) :
C-2. Required point is foot of \(\perp\)
\[
\frac{x - 3}{2} = \frac{y + 1}{-5} = -\left(\frac{6 + 5 + 8}{4 + 25}\right) = -1
\]
\[
x = 1, \quad y = 4
\]
C-4.* Let point on line be
\((h, 4 - 2h)\) (chord of contact)
\[
hx + y (4 - 2h) = 1
\]
\[
h(x - 2y) + 4y - 1 = 0 \quad \text{Point} \left(\frac{1}{2}, \frac{1}{4}\right)
\]
Section (E) :
E-1. Let required circle is \(x^2 + y^2 + 2gx + 2fy + c = 0\)
Hence common chord with \(x^2 + y^2 - 4 = 0\)
is \(2gx + 2fy + c + y = 0\)
This is diameter of circle \(x^2 + y^2 = 4\) hence \(c = -4.\)
Now again common chord with other circle
\[
2x(g + 1) + 2y(f - 3) + (c - 1) = 0
\]
This is diameter of \(x^2 + y^2 - 2x + 6y + 1 = 0\)
\[
2(g + 1) - 6(f - 3) + 5 = 0
\]
\[
2g - 6f + 15 = 0
\]
locus \(2x - 3y - 15 = 0\) which is st. line.
E-2. Common chord of given circle

6x + 4y + (p + q) = 0

This is diameter of \( x^2 + y^2 - 2x + 8y - q = 0 \)

centre \((1, -4)\)

\(6 - 16 + (p + q) = 0\)

\(\Rightarrow p + q = 10\)

\[\text{EXERCISE # 2}\]

\[\text{PART - I}\]

3. Equation of circle whose diameter’s end points are \((a, b)\) and \((h, k)\)

\[(x - a) (x - h) + (y - b) (y - k) = 0\]

\[x^2 + y^2 - x(a + h) - y(b + k) + ah + bk = 0\]

It touches x-axis.

\[\text{Hence } g^2 = c \Rightarrow \left(\frac{a + h}{2}\right)^2 = ah + bk\]

\[\Rightarrow (h - a)^2 = 4bk\]

\[\therefore \text{Locus of } (h, k) \text{ is } (x - a)^2 = 4by.\]

5. As we know if two lines are \(\perp\)

\[m_1 \cdot m_2 = -1\]

\[\left(\begin{array}{c}
\beta - k \\
\alpha - h
\end{array}\right) \left(\begin{array}{c}
\beta \\
\alpha
\end{array}\right) = -1\]

\[\beta^2 - \beta k = -\alpha^2 + \alpha h\]

\[\text{Locus of } (\alpha, \beta) \text{ is } x^2 + y^2 = xh + yk\]

6. \(\left(2h - a, -\frac{b}{2}\right)\) lies on circle

\[2(2h - a)(2h - 2a) - \frac{b}{2} (-2b) = 0\]

\[4(h - a)(2h - a) + b^2 = 0\]

\[8h^2 - 12ah + 4a^2 + b^2 = 0\]

\[D > 0\]

\[144a^2 - 4 \times 8 (4a^2 + b^2) = 0\]

\[9a^2 - 8a^2 - 2b^2 > 0 \Rightarrow a^2 > 2b^2\]

9. Let required equation of circle is \(x^2 + y^2 + 2gx + 2fy + c = 0\)

Now common chord of given circle with required circle are

Common chord \(2gx + 2fy + (c + 4) = 0\) it is also diameter of circle \(x^2 + y^2 = 4\). Hence \(c = -4\)

Similarly with \(x^2 + y^2 - 6x - 8y + 10 = 0 \Rightarrow 2x(g + 3) + 2y(f + 4) - 14 = 0\)

\[\Rightarrow 6(g + 3) + 8(f + 4) - 14 = 0\]
\[ 6g + 8f + 36 = 0 \]
\[ 3g + 4f + 18 = 0 \]

With circle \( x^2 + y^2 + 2x - 4y - 2 = 0 \)
\[ 2x (g - 1) + 2y(f + 2) - 2 = 0 \]
\[ -2g + 4f + 8 = 0 \]
\[ 2g - 4f - 8 = 0 \]

After simplification \( g = -2, f = -3, c = -4 \)
Hence circle \( x^2 + y^2 - 4x - 6y - 4 = 0 \)

13. \( 4 \ell^2 - 5m^2 + 6 \ell + 1 = 0 \)
\[ \Rightarrow (3 \ell + 1)^2 = 5(\ell^2 + m^2) \]
\[ \Rightarrow \left| \frac{3 \ell + 0 + m + 1}{\sqrt{\ell^2 + m^2}} \right| = \sqrt{5} \]
Hence centre \((3, 0)\), radius = \(\sqrt{5}\)

15. Equation of circle having centre \((x_1, y_1)\) and radius 'd'
\( (x - x_1)^2 + (y - y_1)^2 = d^2 \)
\[ x^2 + y^2 = a^2 \]
Equation of common chord
\[ 2xx_1 + 2yy_1 - x^2 - y^2 - a^2 + d^2 = 0 \]
\[ 2xx_1 + 2yy_1 - 2a^2 + d^2 = 0 \]

**PART - II**

1. Point \( \left( t, \frac{1}{t} \right) \) lies on \( x^2 + y^2 = 16 \)
\[ t^2 + \frac{1}{t^2} = 16 \]
\[ \Rightarrow t^4 - 16t^2 + 1 = 0 \] (i)
If roots are \( t_1, t_2, t_3, t_4 \) then
\[ t_1t_2t_3t_4 = 1 \] (ii)

5. \[ \left| \frac{-1 + 0 + c}{\sqrt{2}} \right| = \sqrt{2} \Rightarrow c - 1 = \pm 2 \Rightarrow c = -1, 3 \]
But \( c = -1 \) common point is one
\( c = 3 \) common point is infinite
Hence \( c = -1 \) is Answer.

8. Equation of chords of contact from \((0, 0)\) & \((g, f)\)
\[ gx + fy + c = 0 \]
\[ gx + fy + g(x + g) + f(y + f) + c = 0 \]
\[ gx + fy + \frac{(g^2 + f^2 + c)}{2} = 0 \]
Distance between these parallel lines
\[ \left| \frac{g^2 + f^2 - c}{2\sqrt{g^2 + f^2}} \right| \]

11. \((2, 1)\)
\((g, -f)\)
\((0, -1)\)
\((x + g)(x - 2) + (y + f)(y - 1) = 0 \)
12. \( \cos \pi/3 = \frac{\sqrt{(h+2)^2 + (k-3)^2}}{5} \)

Locus \((x + 2)^2 + (y - 3)^2 = 6.25\)

14. slope of \(C_1C_2\) is \(\tan \alpha = -\frac{4}{3}\)

By using parameteric coordinates
\(C_2(\pm 3 \cos \alpha, \pm 3 \sin \alpha)\)
\(C_2(\pm (-3/5), \pm (4/5))\)
\(C_2(\pm 9/5, \pm 12/5)\)

20. \((x^2 + y^2 - 6x - 4y - 12) + \lambda(4x + 3y - 6) = 0\)

This is family of circle passing through points of intersection of circle \(x^2 + y^2 - 6x - 4y - 12 = 0\) and line \(4x + 3y - 6 = 0\)

other family will cut this family at A & B.

Hence locus of centre of circle of other family is this common chord \(4x + 3y - 6 = 0\)

22. Let any point \(P(x_1, y_1)\) to the circle \(x^2 + y^2 - \frac{16x}{5} + \frac{64y}{15} = 0\)

\(x_1^2 + y_1^2 - \frac{16}{5}x_1 + \frac{64y}{15} = 0\)

Length of tangent from \(P(x_1, y_1)\) to the circle are in ration

\[
\frac{\sqrt{S_1}}{\sqrt{S_2}} = \frac{\sqrt{x_1^2 + y_1^2 - \frac{24}{5}x_1 + \frac{32}{5}y_1 + 15}}{\sqrt{x_1^2 + y_1^2 - \frac{48}{5}x_1 + \frac{64}{5}y_1 + 60}}
\]

\[
= \frac{\frac{16}{5}x_1 - \frac{64}{15}y_1 - \frac{24}{5}x_1 + \frac{32}{5}y_1 + 15}{\frac{16}{5}x_1 - \frac{64}{15}y_1 - \frac{48}{5}x_1 + \frac{64}{5}y_1 + 60}
\]

\[
= \frac{-24x_1 + 32y_1 + 225}{-96x_1 + 128y_1 + 900}
\]

\[
= \frac{-24x_1 + 32y_1 + 225}{4(-24x_1 + 32y_1 + 225)} = \frac{1}{2}
\]

24. Two fixed pts. are point of intersection of \(x^2 + y^2 - 2x - 2 = 0\) & \(y = 0\)
Point $x^2 - 2x - 2 = 0$

$(x - 1)^2 - 3 = 0$

$\Rightarrow x - 1 = \sqrt{3}, \quad x - 1 = -\sqrt{3}$

$(1 + \sqrt{3}, 0) \quad (1 - \sqrt{3}, 0)$

25. $\left| \frac{4C + 3C - 12}{5} \right| = C \Rightarrow C = 1, 6$

**EXERCISE # 3**

**Match the column :**

1. (A) $S_1 - S_2 = 0$ is the required common chord i.e $2x = a$
   
   Make homogeneous, we get $x^2 + y^2 - 8.4 \dfrac{x^2}{a^2} = 0$
   
   As pair of lines subtending angle of $90^\circ$ at origin
   
   $\therefore$ coefficient of $x^2$ + coefficient of $y^2 = 0$
   $\therefore a = \pm 4$

(B) $y = 22\sqrt{3} (x - 1)$ passes through centre $(1, 0)$ of circle

(C) Three lines are parallel

(D) $2(r_1 + r_2) = 4$

2. $r_1 + r_2 = 2$

3. $\dfrac{r_1 + r_2}{2} = 1$

**Comprehension # 2 (6 to 8)**

6. $\triangle PQC_1$ and $\triangle PRC_2$ are similar

7. Let mid point $m(h, k)$. Now equation of chord $T = S_1$

   $hx + ky + 3(x + h) = h^2 + k^2 + 6h$

   it passes through $(1, 0)$

   $h + 3(1 + h) = h^2 + k^2 + 6h$

   locus $x^2 + y^2 + 2x - 3 = 0$

   But clear from Geometry it will be arc of BC

8. Common chord of $S_1$ & answer of 7

   $4x + 3 = 0 \Rightarrow x = -3/4$
at \( x = -3/4 \) \[ \left(-\frac{3}{4} + 3\right)^2 + y^2 = 9 \Rightarrow y^2 = 9 - \frac{81}{16} \]

\[ y^2 = \frac{63}{16} \Rightarrow y = \pm \frac{3\sqrt{7}}{4} \]

Hence \( \tan \theta = \frac{\frac{3\sqrt{7}}{4}}{\left(1 + 3/4\right)} = \frac{3\sqrt{7}}{7} \Rightarrow \tan \theta = \frac{3}{\sqrt{7}} \)

10. Statement-1 is true and statement-2 is false as radius = \[ \frac{1}{2} \sqrt{\alpha^2 + \beta^2} \]

11. Statement-1 : There is exactly one circle whose centre is the radical centre and the radius equal to the length of tangent drawn from the radical centre to any of the given circles.

Statement-2 is True But does not explain Statement-1.

13. \((0, 0) \& (8, 6)\) lie on the director circle of \( x^2 + y^2 - 14x + 2y + 25 = 0 \)

so \( \alpha - \beta = 0 \)

16. \[ P = \frac{6 + 5 + 18}{\sqrt{29}} = \sqrt{29} \]

\[ r^2 = p^2 + 3^2 = 38 \Rightarrow r = \sqrt{38} \]

19. \( x^2 + y^2 - 8x - 12y + p = 0 \)

Power of \((2, 5)\) is \( S_1 = 4 + 25 - 16 - 60 + P = P - 47 < 0 \Rightarrow P > 47 \)

Circle neither touches nor cuts coordinate axes

\[ g^2 - c < 0 \Rightarrow 16 - p < 0 \Rightarrow p > 16 \]

\[ f^2 - c < 0 \Rightarrow 36 - p < 0 \Rightarrow p > 36 \]

taking intersection \( P \in (36, 47) \)

**EXERCISE # 4**

**PART - I**

1. The lines given by \( x^2 - 8x + 12 = 0 \) are \( x = 2 \) and \( x = 6 \).

The lines given by \( y^2 - 14y + 45 = 0 \) are \( y = 5 \) and \( y = 9 \)

Centre of the required circle is the centre of the square.

\( \therefore \) Required centre is

\[ \left(\frac{2 + 6}{2}, \frac{5 + 9}{2}\right) = (4, 7) \).

2. Clearly from the figure the radius of bigger circle

\[ r^2 = 2^2 + \{(2 - 1)^2 + (1 - 3)^2\} \]

\[ r^2 = 9 \text{ or } r = 3 \]

3. The equation of circle having tangent \( 2x + 3y + 1 = 0 \) at \( (1, -1) \)

\[ \Rightarrow (x - 1)^2 + (y + 1)^2 + \lambda(2x + 3y + 1) = 0 \]

\[ x^2 + y^2 + 2x(\lambda - 1) + y(3\lambda + 2) + (\lambda + 2) = 0 \quad \text{... (i)} \]
equation of circle having end points of diameter \((0, -1)\) and \((-2, 3)\) is
\[x(x + 2) + (y + 1)(y - 3) = 0\]
or
\[x^2 + y^2 + 2x - 2y - 3 = 0\]
since (i) & (ii) cut orthogonally
\[
\therefore \frac{2(2\lambda - 2)}{2} + \frac{2(3\lambda + 2)}{2}(-1) = \lambda + 2 - 3
\]
\[\Rightarrow 2\lambda - 2 - 3\lambda - 2 = \lambda - 1\]
\[\Rightarrow 2\lambda = -3\]
\[\Rightarrow \lambda = -3/2\]
\[
\therefore \text{from equation (i), equation of required circle is}
\]
\[2x^2 + 2y^2 - 10x - 6y + 1 = 0\]

4. \[
\sqrt{(h - 0)^2 + (k - 1)^2} = 1 + |k|
\]
\[
\Rightarrow h^2 + k^2 - 2k + 1 = 1 + 2|k| + k^2
\]
\[
\Rightarrow h^2 = 2|k| + 2k
\]
\[
\Rightarrow x^2 = 4y \text{ if } y \geq 0 \text{ & } x = 0 \text{ if } y \leq 0
\]

5. Clearly \(P\) is the incentre of triangle \(ABC\).
\[
r = \frac{s}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}
\]
Here \(2s = 7 + 8 + 9 \Rightarrow s = 12\)
Here \(r = \frac{5.4.3}{12} = \sqrt{5}\)

6. Statement-1 is true because point \((17, 7)\) lies on the director circle and Statement-2 is equation of director circle of given circle.

7. \(18 = \frac{1}{2}(3\alpha)(2r) \Rightarrow \alpha r = 6\)

Line, \(y = \frac{-2r}{\alpha} (x - 2\alpha)\) is tangent to circle
\[(x - r)^2 + (y - r)^2 = r^2\]
\[2\alpha = 3r \text{ and } \alpha r = 6\]
\[r = 2\]

8. \((ax^2 + by^2 + c) (x^2 - 5xy + 6y^2) = 0 \Rightarrow x = 3y \text{ or } x = 2y \text{ or } ax^2 + by^2 + c = 0\)
If \(a = b\) and \(c\) is of opposite sign, then it will represent a circle
Hence (B) is correct option.

9*. \(\text{PS} \cdot \text{ST} = \text{QS} \cdot \text{SR}\)
Now \(\text{HM} < \text{GM}\)
\[
\Rightarrow \frac{2}{\frac{1}{\text{PS}} + \frac{1}{\text{ST}}} < \sqrt{\text{PS} \cdot \text{ST}}
\]
\[ \Rightarrow \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \cdot SR}} \Rightarrow \text{B is correct and A is wrong.} \]

Now \( QR = QS + SR \)
Applying AM > GM
\[ \frac{QS + SR}{2} > \sqrt{QS \cdot SR} \]
\[ QR > 2 \sqrt{QS \cdot SR} \]
\[ \frac{4}{QR} < \frac{2}{\sqrt{PS \cdot ST}} \]
\[ \Rightarrow \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{PS \cdot ST}} > \frac{4}{QR} \] \( \therefore \) D is correct and C is wrong
\[ \Rightarrow \text{‘B’ and ‘D’ are correct.} \]

10. Let \( G(\alpha, \beta) \) be the centre of \( C \)
\[ \alpha = \frac{3\sqrt{3}}{2} - 1 \cdot \cos 30 = \sqrt{3} \]
\[ \beta = \frac{3}{2} - 1 \cdot \sin 30 = 1 \]
\[ \therefore \text{equation of } C \text{ is} \]
\[ (x - \sqrt{3})^2 + (y - 1)^2 = 1 \]

11. \( \angle FGD = \angle DGE = 120^\circ \) \Rightarrow \( F = (\sqrt{3}, 0) \) and \( GF = GE = GD = 1 \)
\[ E = \left( \frac{\sqrt{3}}{2}, \frac{3}{2} \right) \]

12. Slope QR = \( \sqrt{3} \) \( \therefore \) equation of QR is \( y - \frac{3}{2} = \sqrt{3} \left( x - \frac{\sqrt{3}}{2} \right) \)
\[ y = \sqrt{3} x \] and slope of RP = 0
\[ \therefore \) equation RP is \( y = 0 \)

13. The distance between \( L_1 \) and \( L_2 \) is \( \frac{6}{\sqrt{13}} < 2 \)
Statement ‘1’ is True because distance between lines is less than radius but \( L_2 \) need not be a diameter.
Statement ‘2’ is False because if \( L_1 \) is diameter then \( L_2 \) has to be a chord of circle
Thus ‘C’ is correct
14. For required circle, P(1, 8) and O(3, 2) will be the end points of its diameter.

\[ (x - 1)(x - 3) + (y - 8)(y - 2) = 0 \implies x^2 + y^2 - 4x - 10y + 19 = 0 \]

15. \[(r + 1)^2 = \alpha^2 + 9 \]
\[r^2 + 8 = \alpha^2 \]
\[\implies r^2 + 2r + 1 = r^2 + 8 + 9 \]
\[2r = 16 \]
\[r = 8 \]

16. Since distance between parallel chords is greater than radius, therefore both chords lie on opposite side of centre.

\[
2 \cos \frac{\pi}{2k} + 2 \cos \frac{\pi}{k} = \sqrt{3} + 1
\]

Let \( \frac{\pi}{2k} = 0 \)

\[
\implies 2 \cos \theta + 2 \cos 20 = \sqrt{3} + 1 \]
\[\implies 2 \cos \theta + 2 (2 \cos^2 \theta - 1) = \sqrt{3} + 1 \implies 4 \cos^2 \theta + 2 \cos \theta - (3 + \sqrt{3}) = 0 \]
\[
\implies \cos \frac{\pi}{2k} = \frac{\sqrt{3}}{2}, \frac{-(\sqrt{3} + 1)}{2} \text{ rejected}
\]
\[\implies \frac{\pi}{2k} = \frac{\pi}{6} \implies k = 3 \implies [k] = 3 \]

17. Let equation of circle is
\[x^2 + y^2 + 2gx + 2fy + c = 0 \]
as it passes through (-1,0) & (0,2)
\[\therefore 1 - 2g + c = 0 \text{ and } 4 + 4f + c = 0 \]
also \(f^2 = c\)
\[\implies f = -2, c = 4 \text{; } g = \frac{5}{2} \]
\[\therefore \text{equation of circle is} \]
\[x^2 + y^2 + 5x - 4y + 4 = 0 \]
which passes through (-4, 0)

18. \[2x - 3y = 1, x^2 + y^2 \leq 6 \]

\[
S = \left\{ \left( \frac{2}{4}, \frac{3}{4} \right), \left( \frac{5}{4}, \frac{3}{4} \right), \left( \frac{1}{4}, -\frac{1}{4} \right), \left( \frac{1}{4}, \frac{1}{4} \right) \right\}
\]

(I) (II) (III) (IV)

Plot the two curves
I, III, IV will lie inside the circle and point (I, III, IV) will lie on the P region if (0, 0) and the given point will lie opposite to the line \(2x - 3y - 1 = 0\).

\[P(0, 0) = \text{negative, } P\left( \frac{2}{4}, \frac{3}{4} \right) = \text{positive, } P\left( \frac{1}{4}, -\frac{1}{4} \right) = \text{positive, } P\left( \frac{1}{8}, \frac{1}{4} \right) = \text{negative} \]
P \left( \frac{5}{2}, 4 \right) = \text{positive, but it will not lie in the given circle}

so point \( \left( \frac{3}{4}, \frac{3}{4} \right) \) and \( \left( \frac{1}{4}, -\frac{1}{4} \right) \) will lie on the opp side of the line

so two point \( \left( \frac{3}{4}, \frac{3}{4} \right) \) and \( \left( \frac{1}{4}, -\frac{1}{4} \right) \)

Further \( \left( \frac{3}{4}, \frac{3}{4} \right) \) and \( \left( \frac{1}{4}, -\frac{1}{4} \right) \) satisfy \( S_1 < 0 \)

19. Circle \( x^2 + y^2 = 9 \)
line \( 4x - 5y = 20 \)

\( P \left( t, \frac{4t - 20}{5} \right) \)

equation of chord AB whose mid point is \( M(h, k) \)
\[ T = S_1 \]
\[ \therefore hx + ky = h^2 + k^2 \]

equation of chord of contact AB with respect to P.
\[ T = 0 \]
\[ tx + \left( \frac{4t - 20}{5} \right)y = 9 \]

comparing equation (1) and (2)
\[ \frac{h}{t} = \frac{5k}{4t - 20} = \frac{h^2 + k^2}{9} \]

on solving \( 45k = 36h - 20h^2 - 20k^2 \) \[ \Rightarrow \quad \text{Locus is} \quad 20(x^2 + y^2) - 36x + 45y = 0 \]

Sol. 20 to 21

20.

B divides \( C_1, C_2 \) in 2 : 1 externally
\[ \therefore \ B(6, 0) \]

Hence let equation of common tangent is
\[ y - 0 = m(x - 6) \]
\[ mx - y - 6m = 0 \]

length of \( \bot \) dropped from center \( (0, 0) \) = radius
\[ \frac{6m}{\sqrt{1 + m^2}} = 2 \Rightarrow m = \pm \frac{1}{2\sqrt{2}} \]

\[ \therefore \quad \text{equation is} \quad x + \boxed{y} = 6 \quad \text{or} \quad x - \boxed{y} = 6 \]

21. Equation of L is
\[ x - \boxed{y} + c = 0 \]

length of perpendicular dropped from centre = radius of circle
\[\therefore \quad \frac{1}{4} = 1 \quad \Rightarrow \quad C = -1, -5\]

\[\therefore \quad x - \frac{y}{1} = 1 \text{ or } x - \frac{y}{5} = 0\]

**PART - II**

1. \[S_1 : (x - 1)^2 + (y - 3)^2 = r^2 \quad \Rightarrow \quad C_1 (1, 3), r_1 = r\]
   \[S_2 : x^2 + y^2 - 8x + 2y + 8 = 0 \quad \Rightarrow \quad C_2 (4, -1), r_2 = 3\]
   circles intersect \[|r_1 - r_2| < C_1, C_2 < r_1 + r_2\]
   \[|r - 3| < 5 \Rightarrow -5 < r - 3 < 5 \Rightarrow -2 < r < 8\]
   After intersection \[2 < r < 8\].

2. \[\text{Point of intersection of } 2x - 3y = 5\]
   \[3x - 4y = 7 \text{ is } (1, -1)\]
   Hence centre \((1, -1), \text{ Area } = 154 = \quad \Rightarrow r = 7\]
   equation of circle \((x - 1)^2 + (y + 1)^2 = 7^2\]
   \[\Rightarrow x^2 + y^2 - 2x + 2y = 47.\]

3. \[\text{Let centre of circle is } (h, k) \text{ and it passes through } (a, b)\]
   equation of circle is \((x - h)^2 + (y - k)^2 = (h - a)^2 + (k - b)^2\]
   This circle cuts \[x^2 + y^2 - 4 = 0\] orthogonally
   \[2g_1g_2 + 2f_1f_2 = c_1 + c_2\]
   \[\Rightarrow \quad g_1(0) + 2f_1(0) = -(h - a)^2 - (k - b)^2 + h^2 + k^2 - 4\]
   \[\Rightarrow \quad 2ah + 2kb = \quad \Rightarrow \quad 2ax + 2by - \quad \Rightarrow \quad 0.\]
   Hence locus of \((h, k)\) is \(2ax + 2by - \quad \Rightarrow \quad 0.\]

4. \[\text{Equation of circle}\]
   \[(x - p)(x - h) + (y - q)(y - k) = 0\]
   \[\Rightarrow x^2 + y^2 - x(h + p) - y(q + k) + (ph + qk) = 0\]
   This circle touches x-axis \(g^2 = c\)
   \[\Rightarrow \quad \quad = ph + qk\]
   Locus of \((h, k)\) is \((x - p)^2 = 4qy.\]

5. \[\text{Point of intersection of } 2x + 3y + 1 = 0\]
   \[3x - y - 4 = 0 \text{ is } (1, -1)\]
   and circumference of circle \(= 2\pi r = 10\pi \quad \Rightarrow \quad r = 5\]
   Hence equation of circle \((x - 1)^2 + (y + 1)^2 = 25\]
   \[\Rightarrow \quad x^2 + y^2 - 2x + 2y - 23 = 0.\]

6. \[\text{By family of circle } x^2 + y^2 - 2x + \lambda(x - y) = 0\]
   centre of this circle \(\quad = \quad \Rightarrow \quad \lambda = 1\]
   Hence \[x^2 + y^2 - x - y = 0.\]

7. \[\text{Let } S_1 : x^2 + y^2 + 2ax + cy + a = 0\]
   \[S_2 : x^2 + y^2 - 3ax + dy - 1 = 0\]
   common chord \(S_1 - S_2 = 0 \quad \Rightarrow \quad 5ax + y(c - d) + (a + 1) = 0\]
   given line is \(5x + by - a = 0\)
compare both \[ a = \frac{1}{2} = 1 \]

\[ (i) (ii) (iii) \]

From (i) & (iii) \( a^2 + a + 1 = 0 \) \( \Rightarrow a = \omega, \omega^2 \) no real \( a \).

8. Draw a line parallel to \( x \)-axis at a distance 2 unit.
   Now by definition of parabola
   locus of a point whose distance from a fixed point \((0, 3)\)
   is equal to its distance from a fixed line is a parabola.

9. Point of intersection of lines
   \( 3x - 4y - 7 = 0 \)
   \( 2x - 3y - 5 = 0 \) is \((1, -1)\)
   Area of circle = \( \pi r^2 = 49 \) \( \Rightarrow r = 7 \)
   Hence equation of circle \((x - 1)^2 + (y + 1)^2 = 7^2 \) \( \Rightarrow x^2 + y^2 - 2x + 2y = 47 \)

10. Locus of \((h, k)\) is \( x^2 + y^2 = \)

11. Let equation of circle is \((x - h)^2 + (y - k)^2 = (h + 1)^2 + (k - 1)^2 \)
   it touches \( x \)-axis \( g^2 = c \)
   \( h^2 = 2k - 2h - 2 \) \( \Rightarrow k = \)
   \( k \in \) \( \Rightarrow k \in \)

12. \[ = -1 \Rightarrow h = -3 \]
   \[ = -2 \Rightarrow k = -4 \] Hence \( Q(-3, -4) \).

13. \( S_1 + \lambda S_2 = 0 \) should satisfy \((1, 1)\)
   \((2 + 3 + 7 + 2p - 5) + \lambda (1 + 1 + 2 + 2 - p^2) = 0 \)
   \( \lambda = \)
   \( p^2 \neq 6 \) \( \Rightarrow p \neq \pm \)
   but at \( p = \pm \) the 2nd circle is
   \( x^2 + y^2 + 2x + 2y - 6 = 0 \)
   satisfies \((1, 1)\) and obviously \( P \) and \( Q \)
   so \( p = \pm \) is also acceptable
\[ \lambda \neq -1 \Rightarrow p \neq 1 \Rightarrow 7 + 2p \neq 6 - p^2 \]
\[ p^2 + 2p + 1 \neq 0 \]
\[ p \neq -1 \]

14. 
\[ r = 5 \]
\[ < 5 \]
\[ \Rightarrow -25 < m + 10 < 25 \]
\[ \Rightarrow -35 < m < 15 \]
Hence correct option is (1)

15. 
\[ x^2 + y^2 = ax \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1) \]
\[ \Rightarrow \text{centre } c_1 \quad \text{and radius } r_1 = \]
\[ x^2 + y^2 = c^2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2) \]
\[ \Rightarrow \text{centre } c_2 (0, 0) \text{ and radius } r_2 = c \]
both touch each other iff
\[ |c_1 c_2| = r_1 \pm r_2 \]
\[ = \]
\[ \Rightarrow 4 = |a| c + c^2 \Rightarrow |a| = c \]

16. Circle whose diametric end points are (1, 0) and (0, 1) will be of smallest radius.
\[ (x-1)(x-0) + (y-0)(y-1) = 0 \]
\[ x^2 + y^2 - x - y = 0 \]

17. 
\[ h^2 = (1 - 2)^2 + (h - 3)^2 \]
\[ 0 = 1 - 6h + 9 \]
\[ 6h = 10 \]
\[ h = 1 \]
Now diameter is \(2h = \)

\[ \text{ADVANCE LEVEL PROBLEM} \]
\[ \text{PART - I} \]

1. 
\[ x^2 + y^2 - 5x + 2y - 5 = 0 \]
\[ \Rightarrow + (y + 1)^2 - 5 - 1 = 0 \]
\[ \Rightarrow + (y + 1)^2 = \]
\[ \Rightarrow \text{So the axes are shifted to } \]
\[ \text{New equation of circle must be } x^2 + y^2 = \]
2. Equation of circum circle of triangle OAB \( x^2 + y^2 - ax - by = 0 \).
Equation of tangent at origin \( ax + by = 0 \).

\[
d_1 = \boxed{} \quad \text{and} \quad d_2 = \boxed{}
\]

\[
d_1 + d_2 = \boxed{d} = \text{diameter}
\]

3. Let \( r \) be the radius of new circle

\[
C_1C_2 = \boxed{}
\]

So \( r = 2 \).

Slope of line joining \( C_1 \) and \( C_2 \) i.e. \( \tan \theta = 2 \)

\[
\therefore \quad \text{Equation of line joining } C_1 \text{ and } C_2 \text{ is}
\]

\[
x = 2 \quad \text{and} \quad y = 5 \quad \therefore \quad \text{Centre (2, 5)}
\]

4. Area of ABCD = \( 4 \).

5. Equations of two circles touching both the axes are

\[
x^2 + y^2 - 2c_1 x - 2c_1 y + c_1^2 = 0 \quad \ldots (i)
\]

\[
x^2 + y^2 - 2c_2 x - 2c_2 y + c_2^2 = 0 \quad \ldots (ii)
\]

\[
\therefore \quad \text{(i) & (ii) are orthogonal also} \quad \therefore \quad 2c_1c_2 + 2c_1c_2 = c_1^2 + c_2^2
\]

or \( 6c_1c_2 = (c_1 + c_2)^2 \) \ldots (iii)

Now point \( P(a, b) \) lies over the circle

\[
x^2 + y^2 - 2cx - 2cy + c^2 = 0.
\]

so \( c^2 - 2c(a + b) + a^2 + b^2 = 0 \) \quad \therefore \quad c_1 \text{ & } c_2 \text{ are roots of this equation}

so \( c_1 + c_2 = 2(a + b) \) \ldots (iv)

and \( c_1c_2 = a^2 + b^2 \) \ldots (v)

from (iii), (iv) & (v), we get

\[
6(a^2 + b^2) = 4(a + b)^2.
\]

6. Let two circles are \( S = 0 \) and \( S' = 0 \)

having radius \( r_1 \) and \( r_2 \) respectively.

\[
S' = 1 = \boxed{}
\]

\[
S' = 1 = \boxed{r_1^2} = f_2^2 S_1
\]

\[
\Rightarrow S_1 - \boxed{} = S'_1 = 0 \quad \therefore \quad \text{Locus of } P(h,k)
\]

\[
S - \boxed{} = S' = 0 \quad \text{which represents the equation of a circle.}
\]
7. \[ \tan 60^\circ = \frac{\sqrt{3}}{1} \]
\[ \text{and} \quad \sin 60^\circ = \frac{1}{2} \]
Let coordinates of any point P on the circle be \( P = (r \cos \theta, r \sin \theta) \)
\[ PA^2 = r^2 + (r \sin \theta)^2 \]
\[ PB^2 = (r \cos \theta)^2 + (1 - r \sin \theta)^2 \]
\[ PC^2 = (r \cos \theta + 1)^2 + (r \sin \theta)^2 \]
\[ PD^2 = (r \cos \theta)^2 + (r \sin \theta + 1)^2 \]
\[ PA^2 + PB^2 + PC^2 + PD^2 = 4r^2 + 8 = 11 \]
\[ r = \frac{3}{4} \]

8. \[ \theta = \tan^{-1} \frac{3}{4} \Rightarrow \tan \theta = \frac{3}{4} \]
\[ \sin \theta = \frac{3}{5} \quad \text{and} \quad \cos \theta = \frac{4}{5} \]
\[ A' = (OA \cos \theta, OA \sin \theta) \]
\[ A' = (3, 2) \]
Similarly \( B' = (OB \cos \theta, OB \sin \theta) = (6, 4) \)
Now it can be checked that circles \( C_1 \) and \( C_2 \) touch each other.
Let the point of contact be \( C \).
\[ C = (1, 1) \]
required radical axis is a line perpendicular to \( A'B' \) and passing through point \( C \)
\[ y - 1 = - \frac{4}{3} (x - 5) \]

9. \[ \text{Equation of circle} \quad (x - 2)^2 + (y + 2)^2 + \lambda (x + y) = 0 \quad \ldots \ldots (i) \]
\[ \text{Centre lies on the x-axis} \]
\[ \lambda = -4 \quad \text{put in (i)} \]
\[ \text{equation of circle is} \quad x^2 + y^2 - 8x + 8 = 0 \]
\[ (\alpha, \beta) \text{ lies on it} \]
\[ \beta^2 = - \alpha^2 + 8\alpha - 8 \geq 0 \]
\[ \text{greatest value of} \; \alpha \; \text{is} \; 4 + 2 \]

10. Let \( d \) be the common difference
\[ \text{the radii of the three circles be} \; 1 - 2d, 1 - d, 1 \]
\[ \text{equation of smallest circle is} \; x^2 + y^2 = (1 - 2d)^2 \quad \ldots \ldots (i) \]
\[ y = x + 1 \quad \text{intersect (i) at real and distinct points} \]
\[ x^2 + x + 2d - 2d^2 = 0 \quad \ldots \ldots (ii) \]
\[ D > 0 \quad \Rightarrow \quad 8d^2 - 8d + 1 > 0 \]
\[ d > \frac{1}{2} \; \text{or} \; d < \frac{1}{2} \]
but \( d \) can not be greater than \( \frac{1}{2} \)
\[ d \in \left( \frac{-1}{2}, \frac{1}{2} \right) \]
11. Let the coordinates of P and Q are \((a, 0)\) and \((0, b)\) respectively

\[
\begin{align*}
\text{\therefore equation of PQ is } & bx + ay - ab = 0 \quad \text{.......(i)} \\
\text{\therefore } & a^2 + b^2 = 4r^2 \quad \text{.....(ii)} \\
\text{\therefore OM } & \perp PQ \\
\text{\therefore equation of OM is } & ax - by = 0 \quad \text{.......(iii)}
\end{align*}
\]

Let \(M(h, k)\)

\[
\begin{align*}
\therefore bh + ak - ab = 0 \quad \text{.......(iv)} \quad \text{and} \quad ah - bk = 0 \quad \text{.......(v)}
\end{align*}
\]

On solving equations (iv) and (v), we get

\[
\begin{align*}
a = \quad \text{and } b = \\
\text{put } a \text{ and } b \text{ in (ii), we get}
\end{align*}
\]

\[
\begin{align*}
(h^2 + k^2)^2 (h^2 + k^2) = 4r^2
\end{align*}
\]

\[
\therefore \text{ locus of } M(h, k) \text{ is } (x^2 + y^2)^2 (x^2 + y^2) = 4r^2
\]

12. Equation of circle passing through \((0, 0)\) and \((1, 0)\) is

\[
\begin{align*}
x^2 + y^2 - x + 2fy = 0 \quad \text{.......(i)} \\
x^2 + y^2 = 9 \quad \text{.......(ii)}
\end{align*}
\]

so equation of Radical axis is \(x = 2fy + 9 \quad \text{.......(iii)}\)

line (iii) is also tangent to the circle (ii)

\[
\begin{align*}
\therefore \text{ on solving (ii) } & \text{ & (iii), we get} \\
(1 + 4f^2)y^2 + 36fy + 72 = 0 \quad \text{.......(iv)} \\
\therefore \quad D = 0 \Rightarrow f = \pm \\
\end{align*}
\]

13. \[
\begin{align*}
a \ell^2 - bm^2 + 2/\ell d + 1 = 0 \quad \text{......(1)} \\
\text{and} \quad a + b = d^2 \quad \text{......(2)}
\end{align*}
\]

Put \(a = d^2 - b\) in equation (1), we get

\[
\begin{align*}
(\ell d + 1)^2 = b(\ell^2 + m^2)
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \ell d + 1 = \quad \text{......(3)}
\end{align*}
\]

From (3) we can say that the line \(\ell x + my + 1 = 0\) touches a fixed circle having centre at \((d,0)\) and radius =

\[
\text{PART - II}
\]

1. Let the circumcentre be \(P(h, k)\)

\[
\begin{align*}
\therefore \text{ Equation of AB is}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow a = \\
on solving
\end{align*}
\]

\[
\begin{align*}
\therefore \text{ locus of circumcentre } P(h,k) \text{ is}
\end{align*}
\]

\[
2 (x + y) - a = 
\]

2. \[
\begin{align*}
S_1 = x^2 + y^2 = a^2 \\
S_2 = x^2 + y^2 = b^2 \\
S_3 = x^2 + y^2 = c^2 \\
equation of \(\ell_1\) is \(ax \cos \theta + ay \sin \theta = b^2\) \\
\ell_1 \text{ is tangent to circle } S_3
\end{align*}
\]
c = \quad \Rightarrow ca = b^2 \quad \text{Hence a,b,c are in G.P.}

3. Equation of circle touching y - axis is
\[ x^2 + y^2 + 2gx + 2fy + f^2 = 0 \]
so \[ 25 + 8g + 6f + f^2 = 0 \]
\[ 29 + 4g + 10f + f^2 = 0 \]
solving above two equations, we get \((g, f) = (-2, -3) \& (-10, -11)\).

So equations of circles are \(x^2 + y^2 - 4x - 6y + 9 = 0\) and \(x^2 + y^2 - 20x - 22y + 121 = 0\).

\[
\tan \theta = 1 \quad \Rightarrow \quad \theta = 45^\circ
\]

4. \( \ell_1 = 4x + 3y = 10 \)
\( \ell_2 = 3x - 4y = -5 \)
Let \( \theta \) be the inclination of \( \ell_2 \)

\[
\therefore \quad \tan \theta = 5
\]

\[
\therefore \quad \text{equation of } \ell_2 \text{ in parametric form}
\]

So \( \tan \theta \) is max at \( k = 3 \).
At \( k = 3, \tan \theta = 1 \quad \Rightarrow \quad \theta = 45^\circ \)

5. \( \therefore \quad \text{centre lies over the line } 2x - 2y + 9 = 0 \)

So let coordinate of centre be \( (x, y) \)

Let the radius of circle be 'r'
So equation of circle is

\[
(x - h)^2 + (y - k)^2 = r^2
\]

\[
x^2 + y^2 - 2hx - y(2h + 9) + 2h^2 + 9h - r^2 + \quad = 0
\]

\[
\therefore \quad \text{given circle cuts orthogonally to } x^2 + y^2 = 4
\]

so \( 2h^2 + 9h + \quad - r^2 = 0 \)

so equation of required circle can be written as \(x^2 + y^2 - 2hx - y(2h + 9) + 4 = 0 \)

\[
(x^2 + y^2 - 9y + 4) + h(-2y - 2x) = 0
\]

so this circle always passes through points of intersection of \(x^2 + y^2 - 9y + 4 = 0\) and \(x + y = 0\)

so fixed points are \((-4, 4)\) and \((4, -4)\)

6. Centre of \( C_1 \) lies over angle bisector of \( \ell_1 \& \ell_2 \)
Equations of angle bisectors are
\[ x = 5 \text{ or } y = - \]

Since centre lies in first quadrant
so it should be on \( x = 5 \).
So let centre be \((5, \alpha)\)

\[ 3 = \] \[ \Rightarrow \alpha = 2, - \]

But \( \alpha \neq - \) so \( \alpha = 2 \).
So equation of circle \( C_2 \) is
\[
(x - 5)^2 + (y - 2)^2 = 5^2
\]

\[ x^2 + y^2 - 10x - 4y + 4 = 0. \]

7. \( OA = a \) and \( AQ = QP = QR \)

\[ \therefore QA = \] \[ \Rightarrow AQ = \] \[ \therefore (OA)^2 = (OQ)^2 + (AQ)^2 \]

\[ a^2 = \alpha^2 + \beta^2 + (p - \alpha)^2 + (q - \beta)^2 \]

\[ 2\alpha^2 + 2\beta^2 - 2p\alpha - 2q\beta + p^2 + q^2 - a^2 = 0. \]

Locus of the middle point \( Q(\alpha, \beta) \) is
\[
2x^2 + 2y^2 - 2px - 2qy + p^2 + q^2 - a^2 = 0
\]

8. Let the equation of required straight line be \( y = mx + c \).

\[ \Rightarrow m = \] \[ \therefore \]

For \( \triangle PCM \)  
\[ = \tan 2\alpha. \]

\[ \Rightarrow PM = 5\cot 2\alpha \] \[ \Rightarrow \]

For \( \triangle PQM \)  
\[ = PM \sin (90 - \alpha) \]

\[ \Rightarrow \]

on solving, we get \( \alpha = 30^\circ \)  
Equation of tangent at \( P(-2, -2) \) is
\[ 3x + 4y + 14 = 0. \]

\[ \tan 60^\circ = \]

\[ \Rightarrow m = \]

Now on substituting value of 'm' in equation (i), we get
\[ c = \] \[ or \]

but \( c \) should be (-ve)

So equation of line \( y = \) \[ x + \]
9. Let the centre of the circle be \((h, k)\) and radius equal to ‘r’
\[ h^2 + k^2 = r^2 \quad \ldots \ldots (i) \]
and \[2 - h - k = r \quad \ldots \ldots (ii)\]
\[ h = 1 - r \quad \ldots \ldots (iii)\]
put \(h = 1 - r\) in \((ii)\), we get \(k = r (1 - \square) + 1\)
Now put the values of \(h\) and \(k\) in \((i)\), we get
\[(r (1 - \square) + 1)^2 + (1 - r)^2 = r^2 \]
hence radius i.e. \(r\) is the root of the equation
\[(3 - 2\square)^2 - 2\square r + 2 = 0\]

10. Let the equation of the circles be \(x^2 + y^2 + 2gx + 2fy + d = 0\) \ldots \ldots (i)
\[ a^2 + 2fa + d = 0 \quad \ldots \ldots (ii)\]
and \[a^2 - 2fa + d = 0 \quad \ldots \ldots (iii)\]
solving \((ii)\) and \((iii)\), we get \(f = 0, d = -a^2\)
put these value of \(f\) and \(d\) in \((i)\), we get
\[x^2 + y^2 + 2gx - a^2 = 0 \quad \ldots \ldots (iv)\]
\[y = mx + c\] touch these circles
\[g^2 + (2cm) g + a^2 (1 + m^2) - c^2 = 0 \quad \ldots \ldots (v)\]
equation \((v)\) is quadratic in ‘\(g\)’
\[g_1\text{ and }g_2\text{ are its two roots}\]
\[g_1g_2 = a^2 (1 + m^2) - c^2 \]
\[\Rightarrow g_1g_2 = -a^2 \]
\[c^2 = a^2 (2 + m^2) \quad \text{Hence proved}\]

11. Let \(\angle OA'B' = \phi\) and \(\angle OAB = \theta\)
\[0 + \phi = \text{ and } \angle OBA = \phi \]
\[\Rightarrow \text{ length of } AB = 'a' \text{ and length of } A'B' = 'b'\]
\[\Rightarrow \text{ from the figure}\]
\[A'(b \cos \phi, 0) \text{ and } A(a \cos \theta, 0)\]
similarly \(B(0, a \sin \theta)\) and \(B'(0, b \sin \phi)\)
Let \((h, k)\) be the centre of circle
\[2h = a \cos \theta + b \cos \phi \quad \ldots \ldots (i)\]
\[\Rightarrow \phi = -\theta \quad \ldots \ldots (ii)\]
and \[2k = a \sin \theta + b \sin \phi \quad \ldots \ldots (i)\]
\[\Rightarrow 2k = a \sin \theta + b \cos \theta \quad \ldots \ldots (ii)\]
on solving \((i)\) and \((ii)\), we get \[\cos \theta = \quad \text{ and } \sin \theta = \quad \text{}\]
\[\Rightarrow \sin^2 \theta + \cos^2 \theta = 1\]
\[\Rightarrow \text{ locus of } C(h, k) \text{ is } (2ax - 2by)^2 + (2bx - 2ay)^2 = (a^2 - b^2)^2\]
12. \[ \therefore \text{One circle lies within the other circle } \Rightarrow C_1C_2 < |r_1 - r_2| \]

\[ \Rightarrow \quad \square < \square \]

squaring both sides, we get

\[ -2gg_1 < -2 \quad \square - 2c \]

\[ \Rightarrow \quad gg_1 > c + \square \cdot \square \]

\[ \Rightarrow \quad gg_1 - c > \square \cdot \square \quad \ldots (i) \]

\[ \Rightarrow \quad gg_1 - c > 0 \quad \Rightarrow \quad gg_1 > c \]

again squaring both sides of (i), we get

\[ -2cgg_1 > -c (g^2 + g_1^2) \]

\[ \Rightarrow \quad c(g - g_1)^2 > 0 \]

\[ \Rightarrow \quad c > 0 \text{ and from (i), we can say that} \]

\[ \therefore \quad gg_1 \text{ will also be } > 0 \]

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**MATHEMATICAL REASONING, INDUCTION & STATISTICS**

**EXERCISE # 1**

**PART - I**

Section (A) :

A-1. By definition of 'statement'.

A-7. The negation of “Everyone in Germany speaks German” is - there is at least one person in Germany who does not speak German.

A-10. Statement (A) All prime numbers are even. Statement (B) All prime numbers are odd. Both false

A-12. If it is a holiday as well as sunday than also the office can be closed.

A-13*. Polygon cannot be both concave and convex

A-16. Obvious

A-19. \((\sim T \square F) \square T \Rightarrow T \quad \therefore \quad (F \square F) \square \Rightarrow T \quad \therefore \quad F \square \Rightarrow T \)

A-21. \(p \Rightarrow q \) means (i) p is sufficient for q (ii) q is necessary for q (iii) p implies q (iv) if p then q (v) p only if q

Section (B) :

B-3. Contrapositive of \((p \square q) \Rightarrow r \sim r \quad \Rightarrow \quad (p \square \hbar) \)

B-4.
Section (C) :

C-2. \[ x_1 + x_2 + \ldots + x_n = nM \]
\[ \Rightarrow (x_1 + x_2 + \ldots + x_n) - x_n + x' = nM - x_n + x' \]
so
\[ \text{average} = \]

C-5. Average speed over the entire distance =
\[ = \]

C-7. \[ \]

C-10. \[ \alpha - \alpha - 3, \alpha - \alpha - 2, \alpha - \alpha + \alpha + 4, \alpha + 5 (\alpha > 0) \]

C-12. \[ = \]
\[ = \]
\[ (2^{n-1}) = \]

C-14 Frequency of \( f = ^{10}C_5 \) which has maximum value

Section (D) :

D-2. 34, 38, 42, 44, 46, 48, 54, 55, 63, 70
\[ \text{median} = \]
\[ = 47 \]
\[ = 13 + 9 + 5 + 3 + 1 + 1 + 7 + 8 + 16 + 23 = 86 \]
so mean deviation about median =
\[ = 8.6 \]

D-5. \[ \sigma_{\text{new}} = \]
\[ = \sigma_{\text{odd}} \]

D-8. \[ = 60 \]
\[ = 10.4 \]
D-9. \[ \sigma = \]  
\[ \sigma_{\text{new}} = \] \[ = \sigma \]

D-11. \[ = 250 \]
\[ \sigma = \] \[ = 5 \]
\[ \text{coeff. of variation} = \] \[ = 10\% \]

Section (E):

E-4*. Let \( n = 1 \) then \( p(A) = 64 \)
Let \( p(k) \) is divisible by 64
\[ 3^{k+2} - 8k - 9 \] is divisible by 64
Now,
\[ P(k + 1) = 3^{2(k + 1) + 2} - 8(k + 1) - 9 \]
\[ = 3^{2k+2} \times 9 - 8 \times k \times 9 - 9 \times 9 - 8 + 72 + 64 k \]
\[ = 9(3^{2k+2} - 8k - 9) + 64 (k + 1) \]
Which is divisible by 64

E-6. Let \( p(n) = n^3 + (n + 1)^3 + (n + 2)^3 \), \( p(A) = 36 \), \( p(B) = 99 \) both are divisible by 99
Let it is true for \( n = k \)
\[ k^3 + (k + 1)^3 + (k + 2)^3 = 9q ; q \in \mathbb{I} \]
adding \( 9k^2 + 27k + 27 \) both sides
\[ k^3 + (k + 1)^3 + (k + 2)^3 + 9k^2 + 27k + 27 = 9q + 9k^2 + 27k + 27 \]
\[ (k + 1)^3 + (k + 2)^3 + (k + 3)^3 = 9r ; r \in \mathbb{I} \]

Comprehension # 1 (1 to 3)
1. If \( p \) then \( q \) means \( p \) only if \( q \)
2. If \( p \) then \( q \) \( \Rightarrow \) \( p \) is sufficient for \( q \)
3. \( p \) is false, \( q \) is false so \( p \to q \) is true.

Comprehension # 2 (4 to 6)
\[ \therefore \] \[ \therefore \ A.M. = \] \[ = 12 \]
\[ \sigma = \] \[ = 3 \]
\[ \text{coeff. of variation} = \] \[ = \times 100 \]

MATCH THE COLUMN
1. \( (A) \) \[ = 5 + \] \[ = 15 \]
\[ M_x = x_i + 10 = 10 + 10 = 20 \]
\[ \text{variance remains unaffected on addition of a constant} \]
(B) \[ 5 + \square = 5 + 10 = 15 \]
\[
M_s = \frac{x_{11}}{n} = 10
\]
(C) Mean and median get multiplied by 2 and variance by \(2^2\)

(D) \[ \square = 16 \]
\[
M_s = \frac{x_{11} + 11}{n} = 10 + 11 = 21
\]

variance remains unaffected on addition of a constant

**EXERCISE # 2**

4.

7.

9. \[
(p \lor q) \lnot [p \lor (p \lnot q)]
\]
\[
= (p \lor q) \lnot [p \lor (p \lnot q)]
\]
\[
= (p \lor q) \lnot [p \lor (p \lnot q)]
\]
\[
= (p \lor q) \lnot (p \lnot q)
\]
\[
= (p \lor q) \lnot (p \lnot q) = t
\]
also \(\lnot p \lor q = t\)

12. \[
(p \lor q) \lor \lnot p = (p \lor q) \lor (q \lor \lnot p)
\]
\[
= t \lor (q \lor \lnot p) = q \lor \lnot p = \lnot p \lor q
\]

15. p: it rains
q: crops will be good
\(S_1: p \rightarrow q\), \(S_2: \lnot p\), \(S: \lnot q\)

Not valid

17. p: it rains tomorrow
q: I shall carry my umbrella
r: cloth is mended
\(P: p \rightarrow (r \rightarrow q)\)
\(Q: p \land \lnot r\)
\(S: \lnot q\)
\(P: T, Q: T\)
\(\therefore S: T\)
\(\therefore S\) not valid
18. S.D.(x) = S.D. (x – 8) = 2

20. n = 200, mean = 40 = 8000

\[ \sigma^2 = (8000 – 34 + 43) = 8009 \]
\[ \text{correct mean} = 40.045 \]

Also \[ \sigma^2 = 225 = 1600 \]
\[ \Rightarrow \Sigma x^2 = 36500 \]
\[ \text{correct} \sigma^2 = 36500 – (34)^2 + (43)^2 = 365693 \]
\[ \text{correct} \sigma^2 = 1828.465 – 1603.603 \]
\[ \sigma = 14.995 \]

21. \[ \sigma^2 = 1.2 \]

so variance of A = 1.2 < 1.25 = variance of B
so more consistent team = A

22. \[ \sigma^2 = 9 \]
\[ \Rightarrow \text{coefficient of variation} = 25 \]

26. (i) Given statement is true for n = 1
(ii) Let us assume that the statement is true for n = k
\[ \text{i.e.} \ 1.3 + 2.3^2 + 3.3^3 + \ldots + k.3^k = \]
(iii) For n = k + 1,
\[ \text{L.H.S.} = 1.3 + 2.3^2 + 3.3^3 + \ldots + k.3^k + (k + 1) 3^{k+1} \]
\[ = \]
\[ \text{R.H.S.} \]
so by principle of mathematical induction the statement is true for all \( n \in \mathbb{N} \)
MATHEMATICAL REASONING:

1. \( r : x \) is a rational number iff \( y \) is a transcendental number
   \[ \Rightarrow r = \sim p \land q \]
   Statement-1 is false and Statement-2 is false.

2. 

3. Statement-1:
   Statement-2: False.

4. Negation of \( \iff Q \) is \( \sim Q \)
   It may also be written as \( \sim Q \)

5. 

6. Let \( p : I \) become a teacher
   \( q : I \) will open a school
   Negation of \( p \implies q \) is \( \sim (p \implies q) = p \land \sim q \)
   i.e. I will become a teacher and I will not open a school.

STATISTICS:

7. Let average marks of the girls = \( x \)
   \[ = 72 \Rightarrow x = 65 \]

8. No change \( \therefore \) median is 5th observation (If observation are in ascending order)
9. Correct variance = \[
= 222 - 144 = 78.00
\]

10. If we change scale by using \( x + h \) then median increases by \( h \). So median is not independent of change of scale. From histogram we can see highest frequency so made.

11. \[
= 0
\]
   \[
= a^2 \Rightarrow \text{S.D.} = |a| = 2
\]

12. so median = 22, mode = 24

13. \[
\sigma^2 \geq 0
\]
   \[
\Rightarrow + 0 \Rightarrow + 0 \Rightarrow n \geq 16
\]

14. Variances remain unaffected by adding some constant to all observations so \( V_A = V_B \) so \( V_A/V_B = 1 \)

15. Let no. of student = 100 number of boys = \( n \),
   \[
= 50 \Rightarrow n = 80
\]
   so 80%

16. \[
= 6 \Rightarrow a + b = 7 \quad \ldots(1)
\]
   \[
= 6.80 \Rightarrow (a - 6)^2 + (b - 6)^2 = 13
\]
solve \( a = 3, b = 4 \)

17. Statement-1 : \[
\Rightarrow (2n + 1 - 3)
\]
   Statement-2 : Obvious

18. \[
= 1 + 50d
\]
   Mean deviation = \[
= 225 \Rightarrow = 225
\]
   \[
\Rightarrow = 255 \Rightarrow d = 10.1
\]
19. \[ \sigma_x^2 = 4 \Rightarrow \sigma_x = 4 \]
\[ \Rightarrow \sigma_x - (2)^2 = 4 \Rightarrow \sigma_x = 8 \]

Similarly \[ \sigma_x = 105 \]

\[ \therefore \sigma^2 = \sigma_x^2 - \sigma_x = 5.5 \]

20. Median = 25.5 a
Mean deviation about median = 50

\[ a + 3a + 5a + \ldots + 49a = 2500 \]
\[ \Rightarrow 50a = 2500 \Rightarrow a = 4 \]

21. Correct mean = observed mean + 2
Correct S.D. = observed S.D. = 2

22. A.M. of \( 2x_1, 2x_2, \ldots, 2x_n \) is

\[ = \frac{2(x_1 + x_2 + \ldots + x_n)}{n} = \frac{2\sigma^2}{n} \]

So statement-2 is false
Variance \( (2x) = 2^2 \) variance \( (x) = 4\sigma^2 \)
so statement-1 is true.

MATHEMATICAL INDUCTION:

23. Put \( k = 1 \)
LHS = 1, RHS = 4
LHR \( \neq \) RHS
Let \( S(k) \) is true
then \( 1 + 3 + 5 + \ldots + (2k - 1) \)
\[ = 3 + k^2 \]
add \( (2k + 1) \) both the side
\[ 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) \]
\[ = 3 + k^2 + 2k + 1 \]
\[ S(k + 1) = 3 + (k + 1)^2 \]
then if \( S(k) \) is true \( S(k + 1) \) is also true.

24. For \( n \geq 2 \)
\[ n^2 + n < n^2 + n + 1 \]
\[ n^2 + n < (n + 1)^2 \]
statement -2 is true
\[ \therefore \frac{n^2}{n} + \frac{n}{n} + \ldots + \frac{1}{n} \geq \frac{n}{n} + \frac{n}{n} + \ldots + \frac{n}{n} \geq n \]
1. Statement $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ are logically equivalent and give same meaning.

2. 

3. 

4. 

5. $p$: Wages will increase  
$q$: there is an inflation  
$r$: cost of living will increase  
$A: p \quad q$  
$B: q \rightarrow r$  
$C: p$  
$S: r$  
$A: T, B: T, C: T \quad \Rightarrow \quad S: T$  
$\therefore S$ valid

6. Here $\sigma^2 = \bar{x} = \frac{8 + 12 + 13 + 15 + 22}{5} = 15.4$  
$\sigma^2 = \frac{64 + 144 + 169 + 225 + 484}{5} = 108.4$  
$\therefore \sigma^2 = \bar{x} - \frac{1}{n} \sum x^2 = \bar{x} - \frac{1}{n} = 21.2$
7. If \( a \leq x \leq b \Rightarrow a \leq x_i \leq b \)
   \( x_i - \frac{x}{n} \leq b - a \)
   \( (x_i - \frac{x}{n})^2 \leq (b - a)^2 \Rightarrow \frac{x}{n}^2 \leq n(b - a)^2 \)
   so \( \text{var}(x) \leq (b - a)^2 \)

8. Total money per kg. = \[ \text{Total money per kg.} = \frac{\text{Total money}}{\text{Total kg}} \]
   so total kg per rupee = \[ \frac{1}{1.92} \]

11. Let \( P(n) \); \( \sin\theta + \sin2\theta + \ldots + \sin n\theta = \sin \theta \)
   \( P(A) \) is true
   Let \( P(k) \) is also true
   \( \sin\theta + \sin2\theta + \ldots + \sin k\theta = \sin \theta \)
   add \( \sin(k + 1)\theta \) both sides
   \( \sin\theta + \sin2\theta + \ldots + \sin k\theta + \sin(k + 1)\theta \)
   \[ = \sin \theta + \sin2\theta + \ldots + \sin k\theta + \sin(k + 1)\theta \]
   \[ = \sin \theta + \sin2\theta + \ldots + \sin k\theta + \sin(k + 1)\theta \]
   \[ = \sin \theta + \sin2\theta + \ldots + \sin k\theta + \sin(k + 1)\theta \]
   \[ \Rightarrow P(k + 1) \) is true

---

**SOLUTION OF TRIANGLE**

**EXERCISE # 1**

**PART - I**

**Section (A)**:

A-1. (i) \( \text{L.H.S.} = a \sin (B - C) + b \sin (C - A) + c \sin (A - B) \)
   \[ = k \sin A \sin (B - C) + k \sin B \sin (C - A) + k \sin C \sin (A - B) \]
   \[ = k (\sin B^2 - \sin^2 C) + k (\sin^2 C - \sin^2 A) + k (\sin^2 A - \sin^2 B) \]
   \[ = 0 = \text{R.H.S.} \]

(ii) \( \text{L.H.S.} = \)
   \[ \text{first term} = \]
   \[ = k^2 \sin (B + C) \sin (B - C) \]
   \[ = k^2 (\sin^2 B - \sin^2 C) \]
   Similarly \( = k^2 (\sin^2 C - \sin^2 A) \)
   and \( = k^2 (\sin^2 A - \sin^2 B) \)
   \[ \therefore \text{L.H.S.} = k^2 (\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B) \]
   \[ = 0 = \text{R.H.S.} \]
(iii) L.H.S. = 2bc \cos A + 2ca \cos B + 2ab \cos C
= b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2
= a^2 + b^2 + c^2
= R.H.S

(iv) L.H.S. = a^2 + b^2 - 2ab \cos C
= a^2 + b^2 - (a^2 + b^2 - c^2)
= c^2 = R.H.S.

(v) \therefore L.H.S. = b^2 \sin 2C + c^2 \sin 2B
= 2b^2 \sin C \cos C + 2c^2 \sin B \cos B
= 2k^2 \sin B \cos B \sin C + 2k^2 \sin C \cos B \cos C
= 2k^2 \sin(B \sin C) \sin(B + C)
= 2bc \sin A

(vi) \therefore R.H.S = c = a \cos B + b \cos A,
b = c \cos A + a \cos C

A-4. \therefore \quad a^2 + b^2 + c^2 = a + c - b
\Rightarrow \sin(B + C) \sin(B - C) = \sin(A + B) \sin(A - B)
\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B
\Rightarrow 2 \sin^2 B = \sin^2 A + \sin^2 C
\Rightarrow 2b^2 = a^2 + c^2 \quad \Rightarrow a^2, b^2, c^2 \text{ are in A.P.}

A-7. \quad \therefore \quad x^2 - Px + Q - R = 0
\therefore \quad a^2 + b^2 + c^2 = P,
a^2b^4 + b^4c^2 + c^4a^2 = Q
\quad a^2b^2c^2 = R \Rightarrow abc = \quad \therefore \quad a^2 + b^2 + c^2 = R

Section (B):

B-1. (i) L.H.S. = 2a \sin^2 \frac{A}{2} + 2 \cos^2 A
= a(1 - \cos c) + c(1 - \cos A)
= a + c - (a \cos C + c \cos A)
= a + c - b
= R.H.S.

(ii) \therefore \quad L.H.S. = a + b + c
= \quad \therefore \quad a + b + c = \quad \therefore \quad a + b + c = \quad \therefore \quad a + b + c = \quad \therefore \quad [a^2 + b^2 + c^2] = \quad \therefore \quad [a^2 + b^2 + c^2] = \quad \therefore \quad [a^2 + b^2 + c^2] =
(iii) \[ \text{L.H.S.} = 2bc(1 + \cos A) + 2ca(1 + \cos B) + 2ab(1 + \cos C) = 2bc + 2ca + 2ab + 2bc \cos A + 2ca \cos B + 2ab \cos C = 2a^2 + b^2 + c^2 = (a + b + c)^2 = \text{R.H.S.} \]

(iv) \[ \therefore \text{L.H.S.} = (b - c) + (c - a) + (a - b) \]

\[ \therefore (b - c) \cot \theta = k(\sin B - \sin C) \]

\[ = 2k \cos \theta \sin \theta \]

\[ = 2k \sin \theta \sin \theta \]

\[ = k[\cos C - \cos B] \]

similarly \[ (c - a) \cot \theta = k[\cos A - \cos C] \]

and \[ (a - b) \cot \theta = k[\cos B - \cos A] \]

\[ \therefore \text{L.H.S.} = k[\cos C - \cos B + \cos A - \cos C + \cos B - \cos A] = 0 = \text{R.H.S.} \]

(v) \[ \text{L.H.S.} = 4\Delta (\cot A + \cot B + \cot C) \]

\[ = 4\Delta \]

\[ a^2 + b^2 + c^2 = \text{R.H.S.} \]

(vi) \[ \text{L.H.S.} = \cos A \cdot \cos B \cdot \cos C = \Delta = \text{R.H.S.} \]

B-3.

Let \( \angle ADB = \theta \)

\[ \therefore \text{we have to prove that} \tan \theta = \]

if we apply \( m - n \) rule, then \( (1 + 1) \cot \theta = 1 \cot C - 1 \cot A. \]

\[ = \]
\[2(a^2 - c^2)\]

\[2\cot \theta = = \tan \theta = \]

**Section (C):**

C-2.

(i) \[r_1 \cdot r_2 \cdot r_3 = = \Delta^2\]

(ii) \[r_1 + r_2 - r_3 + r = 4R \cos C\]

\[\therefore \text{L.H.S.} = = = = = \]

\[= c \]

\[\therefore \cos C = \]

\[\therefore \text{L.H.S.} = = = = = \]

\[= 4R \cos C\]

(iii) \[\therefore \text{L.H.S.} = = = = = \]

\[= [s^2 + (s - a)^2 + (s - b)^2 + (s - c)^2]\]

\[= 4s^2 - 2s(a + b + c) + \Sigma a^2\]

\[= \text{R.H.S.}\]

(iv) \[\therefore \text{L.H.S.} = = = = = \]
\[ (s + s - a + s - b + s - c)^2 = 4 \]

\[ \therefore \text{R.H.S.} = \]

\[ (s - a + s - b + s - c) = \]

\[ (v) \quad \therefore \]

\[ \Delta = 24 \text{ sq. cm} \quad \text{.... (i)} \]

\[ 2s = 24 \quad \Rightarrow \quad s = 12 \quad \text{.... (ii)} \]

\[ \therefore \text{are in A.P.} \]

\[ \therefore \text{are in A.P.} \]

\[ \therefore \text{are in A.P.} \]

\[ \therefore a, b, c \text{ are in A.P.} \quad \Rightarrow \quad 2b = a + c \]

\[ \therefore 2s = 24 \]

\[ \therefore a + b + c = 24 \]

\[ 3b = 24 \]

\[ \therefore b = 8 \quad \Rightarrow \quad a + c = 16 \]

But \[ \Delta = \]

\[ 24 \times 24 = 12 \times (12 - a) \times 4 \times (12 - c) \]

\[ \Rightarrow 12 = 144 - 192 + ac \]

\[ \therefore ac = 60 \text{ and } a + c = 16 \]

\[ \therefore a = 10, c = 6 \quad \text{or} \quad a = 6, c = 10 \text{ and } b = 8 \]

Section (D):

D-1. (i) \[ \alpha = \, , \beta = \, , \gamma = \, \Rightarrow \]

R.H.S. =
PART - II

Section (A) :
A–4. \[ (a + b + c) (b + c – a) = kbc \]  \[ \Rightarrow (b + c)^2 – a^2 = kbc \]
\[ b^2 + c^2 – a^2 = (k – 2) bc \]
\[ \therefore \text{In a } \triangle ABC \quad -1 < \cos A < 1 \]
\[ 0 < k < 4. \]

Section (B) :
B–2. \[ b \cos^2 \frac{A}{2} + a \cos^2 \frac{B}{2} = c. \]
\[ \Rightarrow \sqrt{s – a + s – b} = c \]
\[ \Rightarrow \frac{a + b = 2c}{a, b, c \text{ are in A.P.}} \]

B–5. \[ \Delta = (a + b – c) (a – b + c) \]
\[ \Delta = 4(s – c) (s – b) \]
\[ \therefore \tan A = \cot \text{ ..........(i)} \]

B–6*. (A) \[ \therefore \tan \quad = \cot \text{ ..........(i)} \]
\[ \therefore \tan^2 = \cot^2 = \]
\[ \therefore \quad a = 5 \text{ and } b = 4 \]
\[ \therefore \text{ from equation (i), we get} \]
\[ = \cot \Rightarrow = \cot \Rightarrow = \]
\[ \therefore \cos C = \boxed{\text{ }} = \boxed{\text{ }} = \boxed{\text{ }} = \boxed{\text{ }} \]

\[ \therefore \cos C = \boxed{\text{ }} \Rightarrow c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow c = 6 \]

(B), (C) \[ \therefore \text{Area} = \boxed{\text{ }} \times 5 \times 4 \times \boxed{\text{ }} \]

\[ \text{Area} = \boxed{\text{ }} \text{sq. unit}. \therefore \text{From Sine rule} \]

\[ \boxed{\text{ }} = \boxed{\text{ }} = \boxed{\text{ }} \Rightarrow \sin A = \boxed{\text{ }} = \boxed{\text{ }} \]

\[ \therefore \sin A = \boxed{\text{ }} \]

Section (C):

C-3.

\[ \boxed{\text{ }} = \boxed{\text{ }} \]

\[ = 4 \boxed{\text{ }} \]

\[ = \boxed{\text{ }} \]

C-5\(^*\). (A) \[ \therefore \boxed{\text{ }} + \boxed{\text{ }} + \boxed{\text{ }} \]

\[ = \boxed{\text{ }} + \boxed{\text{ }} + \boxed{\text{ }} \]

\[ = \boxed{\text{ }} \]

(B) \[ \therefore \boxed{\text{ }} + \boxed{\text{ }} + \boxed{\text{ }} \]

\[ = \boxed{\text{ }} + \boxed{\text{ }} + \boxed{\text{ }} = \boxed{\text{ }} \]

(C) \[ \boxed{\text{ }} = \boxed{\text{ }} = \boxed{\text{ }} \Rightarrow \cot A = \cot B = \cot C \]

\[ \Rightarrow A = B = C \]

true for equilateral triangle only

(D) \[ \boxed{\text{ }} = \boxed{\text{ }} = \boxed{\text{ }} \]
\[\cot A = \cot B = \cot C \Rightarrow A = B = C \Rightarrow \text{true for equilateral triangle only}\]

Section (D):

D–1. \[= 2 \Delta\]

D–4*: \[\beta = \cos\]

(A) correct
(B) incorrect

(C) \[= \cos\]

(D) \[= \cos\]

EXERCISE # 2

PART - I

3. If we apply Sine-Rule in \(\Delta ABD\), we get

\[= \Rightarrow AB = \ldots(i)\]

\[\sin \phi = \ldots \text{and } \cos \phi = \ldots\]

\[\therefore \text{from equation (i), we get}\]

\[AB = \ldots \therefore AB = \ldots\]

7. required distance = inradius of \(\Delta ABC\)
\[
\therefore 2s = a + b + b + c + c + a = 2(a + b + c) \]
\[s = a + b + c\]

\[\Delta = \]

\[
\therefore \text{required distance} = \]

\[
= \]

\[
= \]

\[
= \]

8.

(i) L.H.S. = \((r_3 + r_1) (r_3 + r_2) \sin C\)

\[
= \]

\[
= \]

\[
= \]

\[
= 2 sr_3 \]

R.H.S. = \(2r_3\)

\[= 2r_3 = 2sr_3\]

\[
\therefore \text{L.H.S.} = \text{R.H.S.}\]

(ii) L.H.S. =

\[
= \]

\[
= \]

\[
= \]

\[
= \text{R.H.S.}\]

(iii) First term = \((r + r_1) \tan \)

\[
= \]

\[
= \]

\[
= \]

\[
= 0 = \text{R.H.S.}\]

similarly second term = \(c - a\) & third term = \(a - b\)

\[
\therefore \text{L.H.S.} = b - c + c - a + a - b = 0 = \text{R.H.S.}\]
(iv) \[ r_1 + r_2 + r_3 - r = 4R \]
\[ (r_1 + r_2 + r_3 - r)^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2(r_1r_2 + r_2r_3 + r_3r_1) \] ........(i)
\[ r(r_1 + r_2 + r_3) = ab + bc + ca - s^2 \]
and \[ r_1r_2 + r_2r_3 + r_3r_1 = s^2 \]
\[ \therefore \text{from equation (i)} \]
\[ 16R^2 = r^2 + r_1^2 + r_2^2 + r_3^2 - 2(ab + bc + ca - s^2) + 2s^2 \]
\[ r^2 + r_1^2 + r_2^2 + r_3^2 = 16 R^2 - 4 s^2 + 2 (ab + bc + ca) \]
\[ = 16R^2 - (a + b + c)^2 + 2 (ab + bc + ca) \]
\[ = 16R^2 - a^2 - b^2 - c^2 \]

11. (i) EFA is a cyclic quadrilateral

\[ \therefore \ A = 2 \cos A/2 \]
\[ \therefore \ EF = r \cos A/2 \sin A = 2 \cos A/2 \]
similarly \[ \text{DF} = 2 \cos B/2 \text{and DE} = 2 \cos C/2. \]
(ii) ECD is a cyclic quadrilateral

\[ \therefore \ \angle ICE = \angle IDE \]
similarly \[ \angle IDF = \angle IBF \]
\[ \therefore \ \angle FDE = \angle IDE = \angle IDF \]
\[ \text{area of } \triangle DEF = FD \cdot DE \sin \angle FDE \]
\[ = 2r^2 \]
\[ = 2r^2 \]
\[ \text{PART - II} \]

3. \[ \therefore \ ED = -c \cos B \]
\[ = \]
\[ = \]
5. \( f = R \cos A, \quad g = R \cos B, \quad h = R \cos C. \)

\[
\begin{align*}
\sqrt{1 + \frac{1}{4}} + \frac{1}{8} &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
\frac{1}{8} &= \lambda \\
\Rightarrow \lambda &= 8
\end{align*}
\]

\[
\Rightarrow \lambda = 8
\]

9. MINA is a cyclic quadrilateral

\[
\therefore x = R \sin A = 2r \cos A
\]

\[
\therefore y = R \cos A = 2r \cos B
\]

\[
\therefore z = R \sin C = 2r \cos C
\]

\[
\therefore xyz = \lambda = 4r \sin \theta
\]

12. \( r_1 + r_2 = \)

\[
\therefore \Pi (r_1 + r_2) = R \sin A + R \sin B + R \sin C = 4R \sin A \sin B \sin C
\]

\[
\therefore r_1 + r_2 = 4R
\]

14. \( A, C, G \) and \( B \) are cyclic

\[
\therefore \quad BC_1 \cdot BA = BG \cdot BB_1
\]

\[
\therefore c = 2a
\]

\[
\therefore (2c^2 + 2a^2 - b^2)
\]

\[
\Rightarrow c^2 + b^2 = 2a^2
\]
16. \[ a = 1 \quad \Rightarrow \quad 2s = 6 \]

\[ 2s = 2 \]

\[ R = 1 \quad \Rightarrow \quad \sin A = \frac{1}{2} \]

18. \[ \sin C = \leq 1 \quad \Rightarrow \quad \cos (A - B) \geq 1 \]

\[ \Rightarrow \cos (A - B) = 1 \quad \Rightarrow \quad A - B = 0 \quad \Rightarrow \quad A = B \]

\[ \sin C = 1 \quad \Rightarrow \quad C = 90^\circ \]

20. if we apply m-n Rule in \( \triangle ABE \), we get

\[
(1 + 1) \cot \theta = 1 \cdot \cot B - 1 \cdot \cot \theta
\]

\[ 2 \cot \theta = \cot B - \cot \theta \]

\[ 3 \cot \theta = \cot B \]

\[ \tan \theta = 3 \tan B \quad \ldots \ldots \;(1) \]

Similarly, if we apply m-n Rule in \( \triangle ACD \), we get

\[
(1 + 1) \cot (\pi - \theta) = 1 \cdot \cot \theta - 1 \cdot \cot C
\]

\[ \cot C = 3 \cot \theta \quad \Rightarrow \quad \tan \theta = 3 \tan C \quad \ldots \ldots \;(2) \]

form (1) and (2) we can say that

\[ \tan B = \tan C \quad \Rightarrow \quad B = C \]

\[ A + B + C = \pi \]

\[ \Rightarrow \quad A = \pi - (B + C) \]

\[ = \pi - 2B \quad \Rightarrow \quad B = C \]

\[ \therefore \quad \tan A = - \tan 2B \]

\[ \Rightarrow \quad \tan A = \]
22. \[ r_1 - r = \tan \theta = a \tan \theta \]

\[ \because \Pi (r_1 - r) = abc \tan \theta \tan \theta \tan \theta \]

\[ = abc \Pi \tan \theta \]

\[ = abc \]

\[ = 4Rr^2 \]

**EXERCISE # 3**

2. Match the column

(A) \[ AA, \text{ and } BB, \text{ are perpendicular} \]

\[ \therefore a^2 + b^2 = 5c^2 \]

\[ \therefore c^2 = 5 \Rightarrow c = 5 \]

\[ \cos C = \frac{a}{c} = \frac{3}{5} \]

\[ \sin C = \frac{b}{c} = \frac{4}{5} \]

\[ \therefore \Delta = ab \sin C = 24 \]

(B) \[ \because \text{G.M.} \geq \text{H.M.} \]

\[ (r_1, r_2, r_3)^\frac{1}{3} \geq \Rightarrow (r_1, r_2, r_3)^\frac{1}{3} \geq 3r \Rightarrow \]

\[ \Delta^2 = 11 \geq 27 \]

(C) \[ \tan^2 \theta = \frac{a}{b} = \frac{5}{4} \Rightarrow a = 5, b = 4 \]

\[ 2s = 9 + c \]

\[ c^2 = 36 \Rightarrow c = 6 \]

(D) \[ 2a^2 + 4b^2 + c^2 = 4ab + 2ac \Rightarrow (a - 2b)^2 + (a - c)^2 = 0 \Rightarrow a = 2b = c \]

\[ \cos B = \frac{a}{c} = \frac{3}{5} \]

\[ \therefore 8 \cos B = 7 \]
COMPREHENSION # 2 (Q. No. 7 to 10)

7. Clearly

8. Let $\angle I_1I_2I_3 = 0$
Then angle of pedal triangle $= \pi - 2\theta = A$

9. Side of pedal triangle $= I_2I_3\cos\theta = BC$

10. $I_1 = 4R \sin \theta$

11. $I_2 = 4R \cos \theta$

12. $I_1 I_2 = 4R \cos \theta$ if we apply Sine Rule in $\triangle I_1 I_2 I_3$, then

13. $\sin \theta = \frac{r_1}{2R} = \frac{r_2}{2R}$

14. $3 \sin \theta - 4 \sin^3 \theta = $

15. $\Rightarrow r_1 = r_2 \Rightarrow r = a$, cm.

16. $ax^2 + bx + c = 0 \quad \ldots(1)$

17. $x^2 + k = 0 \quad \ldots(2)$

18. roots of (2) are imaginary and $a$, $b$, $c$ are real

19. $\Rightarrow C = $
EXERCISE # 4

PART - I

1. We have \( a^2 \geq a^2 - (b - c)^2 = (a + b - c)(a - b + c) \)
\( \Rightarrow \ a^2 \geq (2s - 2c)(2s - 2b) = 4(s - b)(s - c) \)
similarly \( b^2 \geq 4(s - c)(s - a) \)
and \( c^2 \geq 4(s - a)(s - b) \).
Multiplying the above inequalities, we get
\( a^2b^2c^2 \geq 64(s - a)(s - b)(s - c)^2 \)
\( \Rightarrow \Delta \geq \) ... (1)
Equality occurs if and only if
\( (b - c)^2 = 0 \)
\( (c - a)^2 = 0 \)
and \( (a - b)^2 = 0 \)
i.e if and only if \( a = b = c \).

2. (A) \( a, \sin A, \sin B \) are given one can determine \( b = c \). So the three sides are unique. So option (a) is incorrect option
(B) The three sides can uniquely determine a triangle. So option (b) is incorrect option.
(C) \( a, \sin B, R \) are given one can determine \( b = 2R \sin B \), \( \sin A = \) ... So \( \sin C \) can be determined. Hence side \( c \) can also be uniquely determined
(D) for \( a, \sin A, R \)
\( = 2R \)
But this could not determine the exact values of \( b \) and \( c \)

3. \( I_n = 2n \times \text{area of } \triangle O_1A_1I_1 \)
\( \Rightarrow I_n = 2n \times \text{area of } \triangle A_1I_1O_1 \)
\( \Rightarrow I_n = n \times \sin \frac{1}{2} \times \cos \frac{1}{2} \)
\( \Rightarrow I_n = \sin \frac{1}{2} \times \cos \frac{1}{2} \) .... (1)
\( O_n = 2n \times \text{area of } \triangle O_1B_1O_1 \)
\( \Rightarrow O_n = 2n \times \text{area of } \triangle O_1B_1O_1 = n \times \tan 1 = n \tan 1 \)
\( \Rightarrow O_n = n \tan 1 \) .... (2)

Now \( \text{R.H.S.} = \) ...
\( = \times 2 \cos^2 \)
\( = O_n \cos^2 \)
\( = n \tan \cos^2 = \sin I_n = \text{L.H.S} \)
4. Let angle of the triangle be $4x$, $x$ and $x$.
Then $4x + x + x = 180^\circ$ \implies x = 30^\circ$
Longest side is opposite to the largest angle.
Using the law of sines
\[
\frac{a}{\sin A} = 2R
\]
\[
\therefore a = R, b = R, c = \frac{R}{\sin 30^\circ} = 2R
\]
\[
\therefore 2S = aR
\]
\[
\therefore \frac{1}{2} \cdot a \cdot b \cdot \sin 120^\circ = \frac{1}{2} \cdot 2R \cdot 2R \cdot \frac{\sqrt{3}}{2}
\]

5. Clearly the triangle is right angled. Hence angles are 30º, 60º and 90º are in ratio 1 : 2 : 3

6. Consider

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

\[
\therefore \frac{a}{b} = \frac{b}{c}
\]

7. Clearly $P$ is the incentre of triangle $ABC$.
\[
r = \frac{b + c - a}{2}
\]
Here $2s = 7 + 8 + 9 \Rightarrow s = 12$
Here $r = \frac{b + c - a}{2}
\]

8. $\Delta = \frac{1}{2} \cdot b \cdot b \cdot \sin 120^\circ = \frac{b^2 \sqrt{3}}{2}$............(1)
Also $a = \frac{b^2}{c}$............(2)
and $\Delta = \frac{a + 2b}{2}$ and $s = \frac{a + 2b}{2}$
\[
\therefore \frac{a + 2b}{2} = \frac{b + c - a}{2}
\]
From (1), (2) and (3), we get $\Delta = \frac{b^2 \sqrt{3}}{2}$

9.* We have $\Delta ABC = \Delta ABD + \Delta ACD$
\[
\Rightarrow bc \sin A = c \cdot AD \sin \angle B + b \times AD \sin \angle C
\]
\[
\Rightarrow AD = \frac{bc}{b + c} \sin A
\]
Again $AE = AD \sec A$
AE is HM of \(b\) and \(c\).

\[
\text{EF} = \text{ED} + \text{DF} = 2\text{DE} = 2 \times \text{AD} \times \cos \theta \times \tan \theta = \text{sin} \theta
\]

As \(\text{DE} = \text{DF}\) and \(\text{AD}\) is bisector \(\triangle\ AEF\) is isosceles.

Hence \(A, B, C\) and \(D\) are correct answers.

10. In \(\triangle\ ABC\), by sine rule

\[
\frac{\text{AB}}{\sin \theta} = \frac{\text{AC}}{\sin \theta} = \frac{\text{BC}}{\sin \theta} \implies C = 45^\circ, C' = 135^\circ
\]

When \(C = 45^\circ\) \(\Rightarrow\) \(A = 180^\circ - (45^\circ + 30^\circ) = 105^\circ\)

When \(C' = 135^\circ\) \(\Rightarrow\) \(A = 180^\circ - (135^\circ + 30^\circ) = 15^\circ\)

Area of \(\triangle\ ABC' = \frac{\text{AB} \times \text{AC}' \times \sin \angle BAC'}{2} = \frac{4 \times \sin (15^\circ)}{2} = 2\)

Area of \(\triangle\ ABC = \frac{\text{AB} \times \text{AC} \times \sin A}{2} = \frac{4 \times \sin (105^\circ)}{2} = 2\)

Absolute difference of areas of triangles = \(|2 - 2| = 4\)

12. \(\cos B + \cos C = 4 \sin^2\theta\)

\[
\Rightarrow 2 \sin \theta = 0
\]

\[
\Rightarrow \cos \theta - 2 \cos \theta = 0 \quad \text{as} \quad \sin \theta \neq 0
\]

\[
\Rightarrow - \cos \theta \cos \theta + 3 \sin \theta \sin \theta = 0
\]

\[
\Rightarrow \tan \theta \tan \theta = \frac{3}{1}
\]

\[
\Rightarrow \frac{2s}{3} \Rightarrow \frac{2s}{3a} \Rightarrow b + c = 2a
\]

Locus of \(A\) is an ellipse
11. \[ \sin 2C + \sin 2A = (a \cos C + c \cos A) = 2 \sin B = 2 \sin 60^\circ = \]

12. \[ \cos = \]

\[ = \]

\[ = \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

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PART - II

1. Let \( a = 3x + 4y, b = 4x + 3y \) and \( c = 5x + 5y \) as \( x, y > 0 \), \( c = 5x + 5y \) is the largest side

\[ \therefore \angle C \text{ is the largest angle. Now} \]

\[ \cos C = \]

\[ = \]

\[ \therefore \angle C \text{ is obtuse angle } \Rightarrow \triangle ABC \text{ is obtuse angled.} \]

2. \( r_1 > r_2 > r_3 \) \[ \Rightarrow \]

\[ \Rightarrow s - a < s - b < s - c \Rightarrow -a < -b < -c; \quad \Rightarrow a > b > c \]

3. \[ \tan = \sin = \]

\[ r + R = \]

\[ \Rightarrow r + R = \cot \]

4. \[ a = \]

\[ \Rightarrow \]

\[ \Rightarrow a + b + c = 3b. \]

\[ \Rightarrow a, b, c \text{ are in A.P.} \]

5. \( AD = 4 \)

\[ \therefore AG = \times 4 = \]

\[ \therefore \text{Area of } \triangle ABG = \times AB \times AG \sin 30^\circ \]

\[ \therefore = \times \times = \]

\[ \therefore \sin 60^\circ = \Rightarrow AB = \]

\[ \therefore \text{Area of } \triangle ABC = 3(\text{Area of } \triangle ABG) = \]

6. \[ \cos \beta = \]

\[ = \]

\[ \Rightarrow \beta = 120^\circ \]

7. \( \angle C = \pi/2 \)

\[ r = (s - c) \tan \quad \therefore C = 90^\circ \]

\[ r = s - 2R \]

\[ \therefore 2r + 2R = 2(s - 2R) + 2R. \]

\[ = 2s - 2R \]

\[ = (a + b + c) - \]

\[ = a + b + c - c \]

\[ = a + b \]
8. are in H.P.
   are in A.P. \( \Rightarrow \) a, b, c are in A.P.

9. \( \cos \) = cos

Let \( \cos \) = for some \( n \geq 3, n \in \mathbb{N} \)

As \( \cos \) < \( \cos \) < \( \cos \) \( \Rightarrow \) \( \cos \) < \( \cos \) < \( \cos \) \( \Rightarrow \) \( \cos \) > \( \cos \) > \( \cos \)

\( \Rightarrow \) 3 < \( n \) < 4, which is not possible
so option (2) is the false statement
so it will be the right choice
Hence correct option is (2)

**ADVANCE LEVEL PROBLEMS**

**PART - I**

1. From figure, \( AD = c \sin B \)
   Hence number of triangle is \( 0 \) if \( b < c \sin B \)
   one triangle for \( b = c \sin B \)
   two triangles for \( b > c \sin B \)

2. \( C = 60^\circ \)
   Hence \( c^2 = a^2 + b^2 - ab \)

   \( \Rightarrow \) \( \frac{c^2}{a^2 + b^2 - ab} = \frac{2 \cos}{2 \cos} = 2 \cos \)

3. Using properties of pedal triangle,
   we have
   \( \angle MLN = 180^\circ - 2A \)
   \( \angle LMN = 180^\circ - 2B \)
   \( \angle MNL = 180^\circ - 2C \)

   Hence the required sum = \( \sin2A + \sin2B + \sin2C \)
   = \( 4 \sin A \sin B \sin C \)

4. From figure, we can observe that \( \triangle OGD \) is directly similar to \( \triangle PGA \)

5. \( BD = s - b, CE = s - c \) and \( AF = s - a \)
   Hence \( BD + CE + AF = s \)

6. \( \Rightarrow \)
⇒ \( \cos \theta = \cos \phi \) as \( \cos \theta = \cos \phi \)

⇒ \( A = B \), in either case

7. Using cosine rule in \( \triangle ABO \), we get \( h = \)

8. In \( \triangle ABD \),

\[ h = k = 2R \]

Comprehension # 1

9. \[ b \sin B + c \sin C + a \sin A = \]

\[ \therefore k = 2R \]

10. \[ \cot A + \cot B + \cot C = \]

\[ (b^2 + c^2 - a^2 + a^2 - b^2 + a^2 + b^2 - c^2) = \]

\[ = (b^2 + c^2 + a^2) = \]

\[ = \Delta \]

\[ \therefore k = \Delta \]

11. \[ \Delta = 6 \]

Comprehension # 2 (12 to 14)

12. \[ \therefore PG = AD \]

\[ = ab \sin C \]

\[ = \Delta \] or \( \Delta = \frac{ac \sin B}{b} \)
13. \( \therefore \text{Area of } \triangle GPL = (PL)(PG) \)
and \( \text{Area of } \triangle ALD = (DL)(AD) \therefore PL = DL \) and \( PG = AD \)

\( \therefore \) \( PG = ac \sin B \)

\( = c \sin B \)

14. \( \therefore \text{Area of } \triangle PQR = \text{Area of } \triangle PGQ + \text{Area of } \triangle QGR + \text{Area of } \triangle RGP \ ...(1) \)

\( \therefore \text{Area of } \triangle PGQ = \text{PG.GQ.sin(} \angle \text{PGQ)} \)

\( = AD \times \text{BE sin } (\pi - C) \)

\( = \times \sin C \)

\( = \text{be sin } A \times \text{ac sin } B \times \sin C \)

\( = \sin A \sin B \sin C \)

Similarly Area of \( \triangle QGR = \sin A \sin B \sin C \) \( \text{and Area of } \triangle RGP = \sin A \sin B \sin C \)

\( \therefore \) From equation (1), we get \( \text{Area of } \triangle PQR = (a^2 + b^2 + c^2) \sin A \sin B \sin C \)

15. In \( \triangle CDB \), \( \Rightarrow \)

Also from same triangle \( \Rightarrow BD = \)

16. \( \cos A \cos B + \sin A \sin B \sin C = 1 \)

\( \Rightarrow \ (\cos A - \cos B)^2 + (\sin A - \sin B)^2 + 2\sin A \sin B(1 - \sin C) = 0 \)

\( \Rightarrow \ A = B \) \( \& \ C = 90^\circ \)

\( \therefore \ a : b : c = 1 : 1 : \)

17. We have \( \Rightarrow \ a : b : c = 5 : 4 : 3 \)
18.from figure, \( OO' = ON - O'N = R - \) 
\[ ZO' = ZM + \]
\[ = R\cos A + \]
from \( \triangle OZO' \), using Pythagoras theorem,
we get \( (R - \) \[ = (R\cos A + \] \[ \Rightarrow \]
\[ = \]

PART - II
1. from \( \triangle AB'C \), 
\[ \Rightarrow AB' = 2R\sin(A + \theta) \]
from \( \triangle AC'B \), 
\[ \Rightarrow AC' = 2R\sin(\theta - A) \]
\[ \therefore B'C' = 2R(\sin(A + \theta) - \sin(\theta - A)) \]
\[ = 4R\cos\theta\sin A = 2a\cos \theta \]
similarly \( C'A' = 2b\cos \theta \)
\[ \therefore \text{area} \triangle A'B'C' = 4\cos^2 \theta \Delta \]

2. \[ c^2 - 2bc \cos A + (b^2 - a^2) = 0 \]
c_1 & c_2 are roots of this quadratic equation
Hence \( (c_1 - c_2)^2 + (c_1 + c_2)^2\tan^2 A = 4a^2 \)

3. Area 
\[ = \]
\[ = \]
\[ = \]
\[ = 2Rs \]
\[ = \]
4. We know that \( OA = R, HA = 2R \cos A \) and applying Appolonious theorem to \( \Delta AOH \), we get

\[
2 \cdot (AQ)^2 + 2 \cdot (OQ)^2 = OA^2 + (HA)^2
\]

\[
\Rightarrow 2 \cdot (AQ)^2 = R^2 + 4R^2 \cos^2 A - \boxed{\quad}
\]

5. Using sine rule, diameter of required circle

\[
\Rightarrow \text{radius} = 10
\]

6. L.H.S. = \[
(a^2 \cdot (b + c - a) + b^2 \cdot (c + a - b) + c^2 \cdot (a + b - c))
\]

\[
= 4R\Delta
\]

7. From the parallelogram \( ABA'C, AA' = 2l_1 \), from \( \Delta AA'C, AA' < b + c \)

\[
\Rightarrow 2l_1 < b + c \quad \text{...(1)}
\]

Similarly \( 2l_2 < c + a \) \quad \text{...(2)}

And \( 2l_3 < a + b \) \quad \text{...(3)}

\( (1) + (2) + (3) \) gives \( l_1 + l_2 + l_3 < 2s \)

8. \( \angle ZXY = \boxed{\quad} \)

\[
\Rightarrow \text{Area of triangle} \]

\[
\boxed{\quad}
\]

\( \therefore \text{Area of} \boxed{\quad} \)
\[ \Delta ABC = \frac{1}{2} \text{absinC} = 2R^2 \sin A \sin B \sin C \]

\[ \Rightarrow \text{Area of } \triangle XYZ = 2R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{R}{2r} \Delta \]

9. Drop a perpendicular from the apex P to the base \( \triangle ABC \).
The foot of perpendicular is at circum centre O of \( \triangle ABC \).

Using given data, we get \( BO = R = \frac{21}{2\sqrt{5}} \)

from right angle \( \triangle POB \), we get

\[ h = PO = \sqrt{PB^2 - OB^2} \]
\[ = 8.83 \text{ m} \]

10. from cyclic quadrilateral CQFP, we get

\( \angle CQP = \angle CFP = B \)

from cyclic quadrilateral AQMF, we get

\( \angle FQM = \angle FAM = 90^\circ - B \)

\[ \Rightarrow \angle AQM = 90^\circ + 90^\circ - B = 180^\circ - B \]

\[ \therefore \angle AQM + \angle CQP = 180^\circ \]

\[ \Rightarrow P, Q, M \text{ are collinear} \]

similarly P, Q, N are collinear

hence, P, Q, M, N are collinear